Collective Dynamics in Networks of Phase Oscillators

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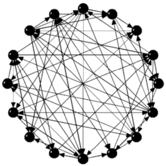
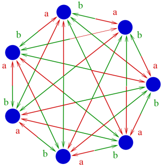
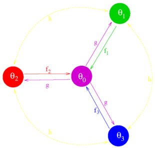
The mathematical definition of the networks of dynamically interacting elements (oscillators, neurons, etc.) always includes the following three main components:

1. description of the *dynamics of an individual element* (not coupled with others) with the help of equations or systems of equations
2. description of the *architecture of interaction between separate elements of the network* that most frequently can be done by using the graph of couplings
3. description of the *the influence of one element upon the other in each link of the coupling* realized with the help of additional terms on the right-hand sides of the system of an individual element or with the use of additional equations or systems of equations (*interaction between nodes*)







By collective dynamics we mean different types of interaction between different elements that have their own individual dynamics.

Collective regimes in phase models:

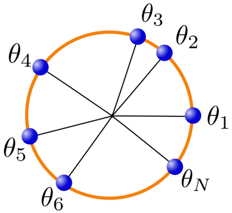
- synchronization
- clustering (partial synchronization)
- anti-phase regimes
- splay states
- slow switching
- chaotic synchronization
- chimera states
- travelling waves
- solitary states
- heteroclinic chimeras
- ...

Generalized Kuramoto model

The system of differential equations describes interactions of N coupled phase oscillators:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\Gamma_{ij}(\theta_i - \theta_j), \quad i = 1, \dots, N, \tag{1}$$

where $\theta_i \in [0, 2\pi) = \mathbb{T}^1$ – phase variables,
ω_i – eigenfrequencies of oscillators,
K_{ij} – parameters (strengths) of couplings between oscillators,
$\Gamma_{ij}(x)$ – smooths 2π–periodic coupling functions.



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Kuramoto model (standard)

$$\Gamma_{ij}(x) = -\sin x, \quad K_{ij} = K$$

Kuramoto-Sakaguchi model

$$\Gamma_{ij}(x) = -\sin(x - \alpha), \quad \alpha \text{ - phase shift parameter}$$

Pikovsky-Rosenblum model

$$\Gamma_{ij}(x) = -\sin(x - \alpha(R, \beta)), \quad \alpha(R, \beta) \text{ - phase shift function with order parameter } R$$

Hansel-Mato-Monier model

$$\Gamma_{ij}(x) = -\sin(x - \alpha) + r\sin(2x - \beta), \quad \alpha, \beta, r \text{ - parameters}$$

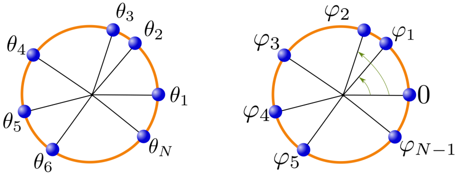
Reduction to phase differences

Generalized Kuramoto model:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\Gamma_{ij}(\theta_i - \theta_j), \quad i = 1, \dots, N, \tag{1}$$

Substitution of variables

$$\varphi_i = \theta_1 - \theta_{i+1}, \qquad i = 1, \dots, N-1. \tag{2}$$



The system in phase differences:

$$\frac{d\varphi_i}{dt} = \Delta_i + \frac{1}{N}\sum_{j=1}^{N}\left(K_{1,j}\Gamma_{1,j}(\varphi_{j-1}) - K_{i+1,j}\Gamma_{i+1,j}(\varphi_{j-1} - \varphi_{i-1})\right), \quad i = 1, \dots, N-1, \tag{3}$$

where $\Delta_i = \omega_1 - \omega_{i+1}$, $i = 1, \dots, N-1$, and $\varphi_0 := 0$.

Definitions

- Complex mean field of (1)

$$Z(t) = R(t)e^{\imath\psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{\imath\theta_j(t)} \tag{4}$$

where $\imath = \sqrt{-1}$, is called the complex order parameter and its amplitude $R(t)$ is called order parameter.

- Two oscillator θ_i and θ_j are phase synchronized if $|\theta_i(t) - \theta_j(t)| \to 0$ as $t \to \infty$.
- θ_i and θ_j are in the antiphase if $|\theta_i(t) - \theta_j(t)| \to \pi$, $t \to \infty$.
- θ_i та θ_j are phase-locked if $|\theta_i(t) - \theta_j(t)| \leq \mathrm{const} < 2\pi \ \forall\ t$.
- θ_i and θ_j desynchronized if they are not phase-locked.
- The system is fully synchronized if all N oscillators are synchronized with each other, i.e. $\Theta_{sync} = (\theta, \dots, \theta)$.
- the system has an m-cluster or partial synchronization if m oscillators are synchronized, $1 < m < N$.
- Oscillators θ_i and θ_j are frequency synchronized if

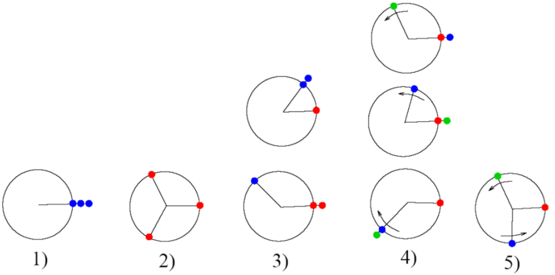
$$\Omega_{ij} := \lim_{t\to\infty} \frac{1}{t} |\theta_i(t) - \theta_j(t)| = 0. \tag{5}$$

Definitions

- The system has the state of global antiphase, if $R = 0$. This regime describes $(N-2)$-dimensional manifold:

$$\mathcal{M}^{(N)} = \left\{ (\theta_1, \ldots, \theta_N) : \quad \sum_{j=1}^{N} \mathrm{e}^{\imath \theta_j} = 0 \right\}. \tag{6}$$

- The system has splay state (or rotating wave) $\Theta^{(k)}_{splay}$ if $\theta_{i+1} = \theta_i + 2k\pi/N$, where $k = 1, \ldots, N-1$, all subscripts are assumed modulo N.
- k-cluster state $\mathcal{P}_k$, $k = 2, \ldots, N-1$, is splitting the set of the oscillators $\Theta = (\theta_1, \ldots, \theta_N)$ into k clusters.
- system (1) has slow switching if the corresponding system in phase differences has a heteroclinic cycle.



Kuramoto model of globally coupled phase oscillators

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} g(\theta_i - \theta_j), \quad j = 1, \dots N, \tag{7}$$

with 2π–periodic smooth coupling function $g(x)$.

The system of globally coupled identical (with $\omega_i = \omega$, $i = 1, \dots, N$) phase oscillators has permutation symmetry $\mathbf{S}_N$. It leads to the existence of cluster invariant manifolds $\mathcal{P}_k$.
Manifolds $\mathcal{P}_{N-1}$ (with $\theta_i = \theta_j$) split phase space $\mathbb{T}^N$ into $(N-1)!$ canonical invariant regions:

$$\mathcal{C} = \{(\theta_1, \dots, \theta_N) : \theta_1 < \theta_2 < \dots < \theta_N < \theta_1 + 2\pi\}. \tag{8}$$

(up to index permutations).

System (7) of identical oscillators has only phase locked solutions.

System (7) of identical oscillators is gradient when $g(x)$ is odd and it is divergence free when $g(x)$ is even.

Globally coupled network of phase oscillators

Kuramoto model of globally coupled phase oscillators

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} g(\theta_i - \theta_j), \quad j = 1, \dots N, \tag{7}$$

with 2π–periodic smooth coupling function $g(x)$.

In the case of identical eigenfrequencies $(\omega_i = \omega)$ the system has special regimes:

1. full synchronization Θ_{sync} which is stable if $g'(x) < 0$;
2. splay state $\Theta^{(k)}_{splay}$ which is stable if

$$\sum_{j=1}^{N-1} g'\left(\frac{2\pi}{N}j\right)\left(1 - \cos\left(\frac{2kj\pi}{N}\right)\right) < 0, \quad k = 1, \dots, N-1; \tag{9}$$

3. antiphase regime $\mathcal{M}^{(N)}$ which is invariant for $N = 2, 3, 4$;
4. two-cluster modes $\Theta_{p,N-p} \subset \mathcal{P}_2$;
5. many-cluster modes if $g(x)$ has two or more harmonics;
6. slow switching that correspond to heteroclinic cycles of the system in phase variables;
7. Chaos for $N \geq 4$ and if $g(x)$ has the third and fourth harmonics [Bick, Timme, Paulikat, Rathlev, Ashwin, PRL, 2011].

Kuramoto model with nonlinear phase shift (Pikovsky-Rosenblum model)

Consider N globally coupled identical phase oscillators:

$$\frac{d\theta_i}{dt} = \omega - \frac{K}{N}\sum_{j=1}^{N}\sin\left(\theta_i - \theta_j - \alpha(R,\beta)\right), \quad i = 1,\ldots,N. \tag{10}$$

where ω, K – main natural frequency and coupling parameter Coupling function

$$g(x) = -\sin(x-\alpha) \tag{11}$$

depends on nonlinear phase shift

$$\alpha = \alpha(R,\beta)$$

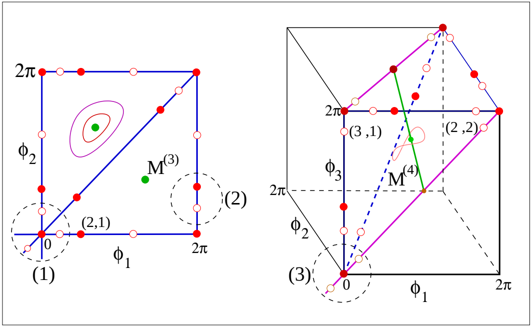
that depends on order parameter R:

$$R = |Z| = \left|\frac{1}{N}\sum_{j=1}^{N} e^{\imath\theta_j}\right| = \frac{1}{N}\sqrt{N + 2\sum_{i,j=1,j\neq j}^{N}\cos(\theta_i - \theta_j)}$$

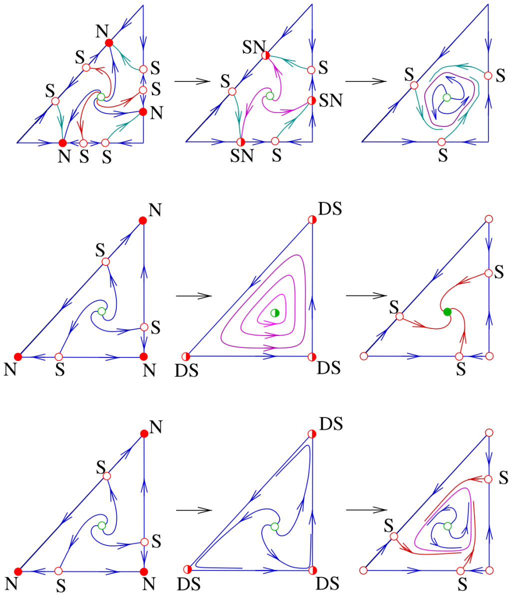
and vector of bifurcation parameters $\beta = (\beta_1,\ldots,\beta_m)$, $m \geq 1$.

The system in phase differences corresponding to (10) with nonlinear phase shift $\alpha(R, \beta) \in C^1(\mathbb{T}^{N-1})$ has special sets:

1. The origin $\Phi_{sync} = (0, \dots, 0)$ which corresponds to regime of the full synchronization Θ_{sync} of the original system (10).
2. $(N-3)$–dimensional invariant antiphase manifold $\mathcal{M}^{(N)}$ that consists of unisolated fixed points.
3. Two-cluste states that blong to invariant manifolds $\mathcal{P}_2$ with isotropi $\mathbf{S}_p \times \mathbf{S}_{N-p}$, $p \neq N/2$.
4. Heteroclinic cycles on the boundaries of invariant regions $\partial\mathcal{C}$.
5. Limit cycles inside invariant regions $\mathcal{C}$ (quasi-periodic for the original system).



Heteroclinic bifurcations of three oscillators



Bifurcation diagram for three coupled oscillators in Pikovsky-Rosenblum model

$$\frac{d\theta_i}{dt} = \omega - \frac{K}{N}\sum_{j=1}^{N} \sin\left(\theta_i - \theta_j - \alpha(R,\beta)\right), \quad i = 1, \dots, N. \tag{10}$$

$$\alpha = \alpha(R,\beta) = \beta_1 + \beta_2 R^2.$$

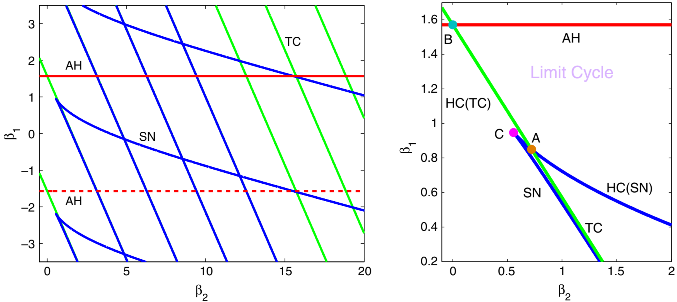
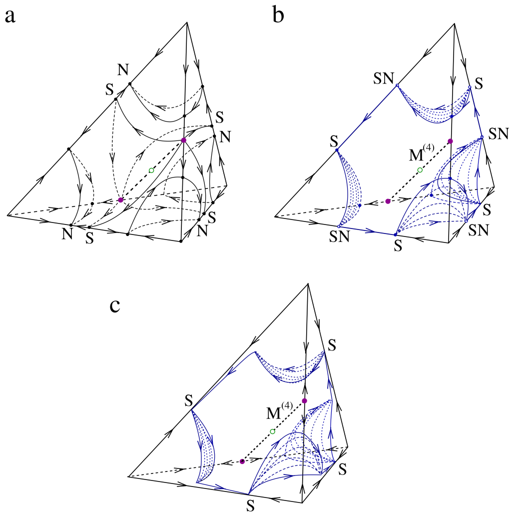


Figure: Bifurcation diagram for $N = 3$ oscillators in the (β_2, β_1) parametric plane. Right panel is an enlargement of the central part of the left one. **TC** — transcritical bifurcation, **SN** — saddle–node bifurcation , **AH** — supercritical Andronov–Hopf bifurcation, **HC(SN)**, **HC(TC)** heteroclinic bifurcations. Points **A**, **B** and **C** are codimension-two bifurcation points. The region where a stable limit cycle exists (right panel) is surrounded by a supercritical AH bifurcation line and two lines of heteroclinic bifurcations of different types.

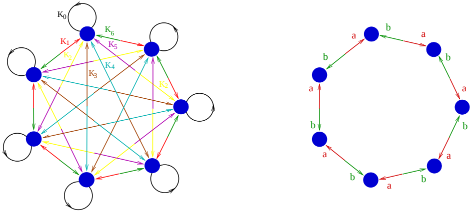
Heteroclinic bifurcation of four oscillators



Translationally invariant ring of coupled phase oscillators with periodic boundary conditions (all subscripts are taken modulo N):

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} K_{j-i}\, g(\theta_i - \theta_j), \quad i = 1, \ldots, N, \tag{12}$$

$$K = \mathrm{circ}(K_0, K_1, \ldots, K_{N-1}) = \begin{pmatrix} K_0 & K_1 & \ldots & K_{N-2} & K_{N-1} \\ K_{N-1} & K_0 & K_1 & \ddots & K_{N-2} \\ \vdots & K_{N-1} & K_0 & \ddots & \vdots \\ K_2 & \ddots & \ddots & \ddots & K_1 \\ K_1 & K_2 & \ldots & K_{N-1} & K_0 \end{pmatrix}$$



Network of seven circularly coupled oscillators: **(Left)** the general case, **(right)** nearest–neighbour coupling: $K_1 = a$, $K_{-1} = b$, $K_i = 0$, $i \neq \pm 1$.

Consider the system in phase differences (12) of three identical phase oscillators with skew–symmetric circulant coupling matrix and Kuramoto–Sakaguchi coupling function:

$$\omega_1 = \omega_2 = \omega_3 \quad \text{or} \quad \Delta_1 = \Delta_2 = 0,$$

$$K = \begin{pmatrix} K_0 & K_1 & K_2 \\ K_2 & K_0 & K_1 \\ K_1 & K_2 & K_0 \end{pmatrix} = \begin{pmatrix} 0 & a & -a \\ -a & 0 & a \\ a & -a & 0 \end{pmatrix}$$

$$g(x) = -\sin(x - \alpha)$$

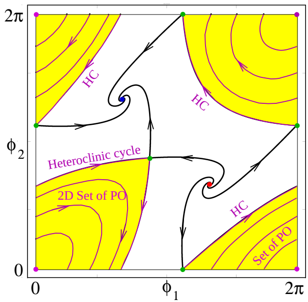
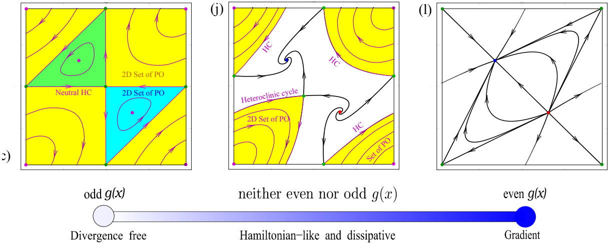


Figure: Phase portrait for $\varphi_1, \varphi_2 \in [0, 2\pi)$ when $\alpha \in (0, \pi/2)$. Yellow – conservative region, white – dissipative region.

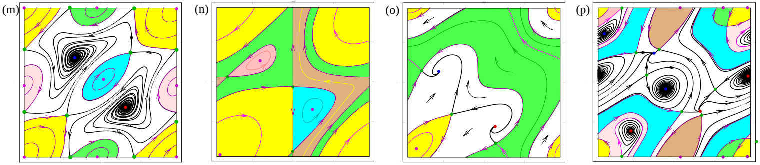
(b1) Non–resonance condition: $(q, \Omega) = \sum_{m=1}^{[(N-1)/2]} q_m \Omega_m \neq 0$ is satisfied for all q with $|q| \leq 2l + 2$ and some $l \in \mathbb{N}$, where $\Omega_m = 2g'(0) \sum_{j=1}^{[(N-1)/2]} K_j \sin\left(\frac{2mj\pi}{N}\right)$.
(b2) Non-degeneracy condition: The leading cubic terms (their imaginary parts) of the normal form are non-degenerate

Theorem. *For skew–symmetric coupling matrix K and $g'(0) \neq 0$ system (12) possesses the following dynamics:*
(A) *Families of periodic orbits in the vicinity of Φ_{sync}: For non–resonance Ω_m, $m = 1, \dots, [(N-1)/2]$, there exists a one–parameter family of periodic solutions $\Phi_\sigma(t)$ in the neighbourhood of Φ_{sync} when N is odd and a two-parameter family $\Phi_{(\sigma_1,\sigma_2)}(t)$ of periodic solutions when N is even, with periods close to $2\pi/\Omega_m$.*
(B) *Dense set of invariant tori in the vicinity of Φ_{sync} for $N \geq 5$: Under conditions (b1) and (b2) in any neighbourhood of Φ_{sync} there exist analytic $[(N-1)/2]$–dimensional quasi–periodic tori with incommensurable frequencies close to $\Omega_1, \dots, \Omega_{[(N-1)/2]}$. The tori are invariant with respect to the flow and with respect to the reversibility transformation $\mathcal{R}$. Moreover, if U_ε is an ε–neighbourhood of Φ_{sync}, then the Lebesgue measure of the invariant tori tends to the full measure of the neighbourhood U_ε, as $\varepsilon \to 0$.*
(C) *Dissipative dynamics: The equilibrium $\Phi_{splay}^{(k)}$, $k \neq 0$, is a sink if the condition $\mathrm{Re}(\lambda_m(\Phi_{splay}^{(k)}) < 0$, $m = 1, \dots, N-1$, is satisfied. In this case $\Phi_{splay}^{(-k)}$ is a source.*

System properties in the skew–symmetric case



(Top) Phase portraits for $\varphi_1, \varphi_2 \in [0, 2\pi)$ in skew–symmetric circulant case. Colour areas indicate conservative regions, white – dissipative.
(bottom) Schematic bifurcation diagram for the skew–symmetric coupling.
The diagram is correct for arbitrary coupling function G and oscillator number N.



Examples of the conservative-dissipative dynamics for three non-identical oscillators with $g(x) = -\sin(x - \alpha) + p\sin(2x)$.

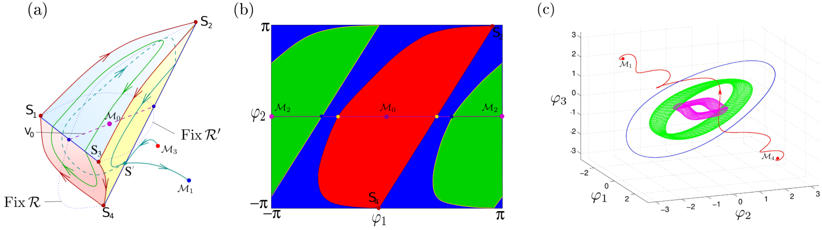


Figure: Dynamics in circulant skew symmetric network for $N = 4$ and $N = 5$ oscillators.
$\mathcal{M}_0 := \Phi_{sync}$ – full synchronization, $\mathcal{M}_i := \Phi^{(i)}_{splay}$ – splay states, $i = 1, \ldots, N-1$.
(a) Structure of boundaries for conservative region for $N = 4$. Conservative region $\mathcal{D}$ is filled with 2–parametric family of periodic orbits and is bounded by a surface of heteroclinic cycles.
(b) Poincare section for $N = 4$ shows attraction basins of conservative regions (red and green) and dissipative region (blue).
(C) Projection of trajectories of (12) for $N = 5$ into subspace $(\varphi_1, \varphi_2, \varphi_3) \in [-\pi, \pi]^3$ of $\mathbb{T}^4$. Blue trajectory is periodic, green and magenta are quasi–periodic with different amplitude, red trajectory is heteroclinic and it connects the repeller $\mathcal{M}_1$ and the attractor $\mathcal{M}_4$.

Coexistence of different solutions in conservative case

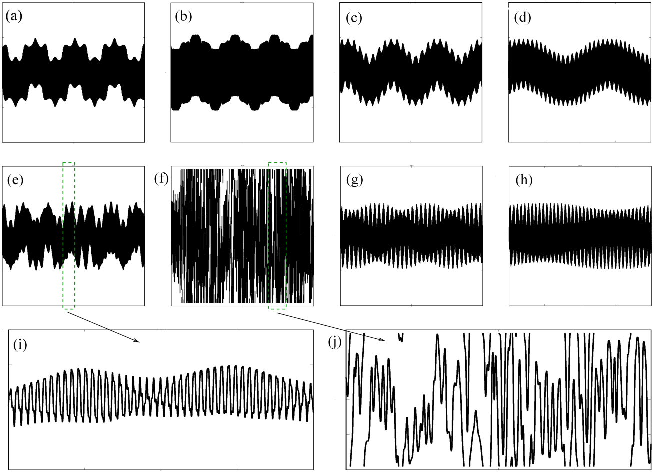


Figure: Coexistence of periodic, quasi-periodic and chaotic solutions for divergence-free system of seven phase oscillators with two couplings from each side. Time-Coordinate: $(t, \varphi_1(t))$, $t \in [0, 2000]$, $\varphi_1 \in [-\pi, \pi]$, для $N = 7$, $K_1 = K_2 = -K_5 = -K_6 = -1$, $K_0 = K_3 = K_4 = 0$, $g(x) = -\sin(x)$. Initial conditions are different in different cases.

Chimera

in Greek mythology

1. (2002) Yoshiki Kuramoto and Dorjsuren Battogtokh: Coherent–Incoherent dynamical regime [1]
2. (2004) Daniel Abrams and Steven Strogatz: name "chimera state" (chimera) [2]
3. (2012) Firs experimental observation of the chimera states [3]

Chimera states appear in Mechanics, Chemistry, Biology, Neuroscience, Electronics, Optics, Sociology, electrochemistry and study of the Financial Market behaviour

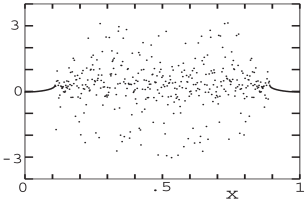
[1] Y. Kuramoto & D. Battogtokh, Coexistence of coherence and incoherence in nonlocally coupled phase oscillators, *Nonlinear Phenomena in Complex Systems*, **5(4)**, 380–385, (2002).

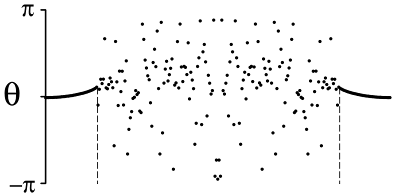
[2] D. Abrams & S. Strogatz, Chimera States for Coupled Oscillators, *Physical Review Lettters*, **93**, 174102, (2004).

[3] A.M. Hagerstrom, T.E. Murphy, R. Roy, P. Hövel, I. Omelchenko and E. Schöll, Experimental observation of chimeras in coupled–map lattices, *Nature Physics*, **8**, 658–661 (2012).

$$\frac{d\theta_i}{dt} = \omega + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\, g(\theta_i - \theta_j), \quad i = 1, \ldots, N, \tag{13}$$

Approximate definition of the chimera state
Regime that for certain choices of parameters and initial conditions, the array would split into two domains: one composed of coherent, phase-locked oscillators, coexisting with other composed of incoherent, drifting oscillators.

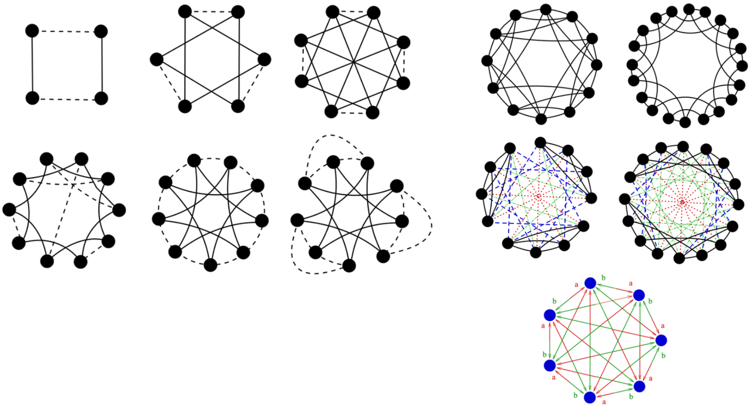




Instantaneous spatial distribution of the phases (snapshot).
One point represents one oscillator $\theta_i \in \mathbb{T}^1$ that runs along its own vertical track. (Left) Kuramoto–Battogtokh, (Right) Abrams–Strogatz.

Indistinguishable phase oscillators

Definition 1
The oscillators are *indistinguishable* if the oscillators are identical and interchangeable in the sense that they have the same number and strength of inputs.

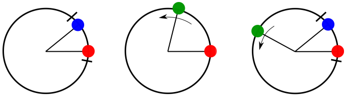


Definition 2
Oscillators i and j on the trajectory of the system (13) are *frequency synchronized* if

$$\Omega_{ij} := \lim_{T\to\infty} \frac{1}{T}[\theta_i(T) - \theta_j(T)] = 0$$

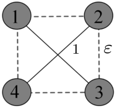
where we chose continuous representation of $\theta_i(t)$, $\theta_j(t)$.

Definition 3 (Weak Chimera [1])
$A \subset \mathbb{T}^N$ is a *weak chimera state* for a coupled phase oscillator system, if it is connected chain–recurrent flow–invariant set such that on each trajectory within A there are i, j and k such that $\Omega_{ij} = 0$ and $\Omega_{ik} \neq 0$.



[1] P. Ashwin & O. Burylko,
Weak chimeras in minimal networks of coupled phase oscillators,
Chaos, **25(1)**, 013106, (2015).

Four oscillators: Stable weak chimera with in–phase and anti–phase groups

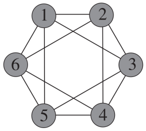


$$
\begin{aligned}
\frac{d\theta_1}{dt} &= \omega + g(\theta_1 - \theta_3) + g(0) + \varepsilon g(\theta_1 - \theta_2),\\
\frac{d\theta_2}{dt} &= \omega + g(\theta_2 - \theta_4) + g(0) + \varepsilon g(\theta_2 - \theta_1),\\
\frac{d\theta_3}{dt} &= \omega + g(\theta_3 - \theta_1) + g(0) + \varepsilon g(\theta_3 - \theta_4),\\
\frac{d\theta_4}{dt} &= \omega + g(\theta_4 - \theta_2) + g(0) + \varepsilon g(\theta_4 - \theta_3),
\end{aligned} \tag{14}
$$

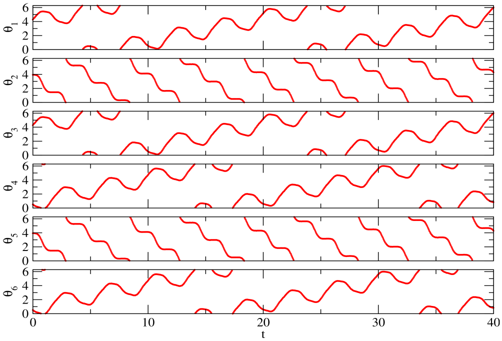
$$
g(\phi) = -\sin(\phi - \alpha) + r\sin(2\phi) \tag{15}
$$

Theorem

There is an open set of (r, α) *such that the four–oscillator system (14), (15) has an attracting weak chimera state for* $\varepsilon = 0$ *that persists for all* ε *with* $|\varepsilon|$ *sufficiently small.*

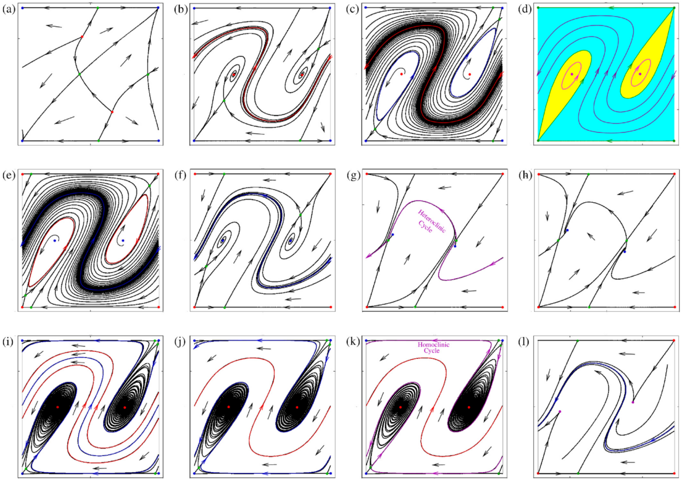


$$\frac{d\theta_i}{dt} = \omega + \frac{1}{N} \sum_{|i-j|=1,2} g(\theta_i - \theta_j), \quad i = 1, \ldots, 6,$$

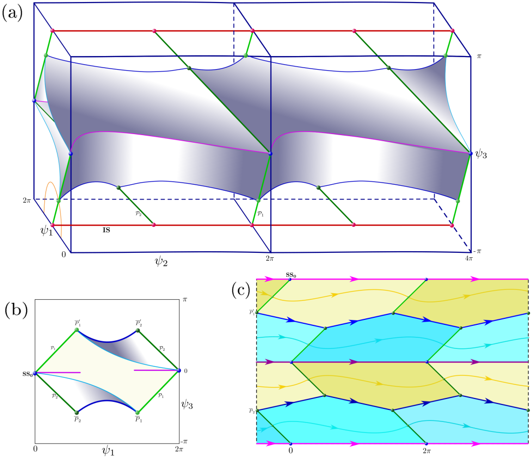


A stable weak chimera state in the ring of six phase oscillators showing for $\alpha = 1.56$, $r = -0.1$. The solution belong to invariant subspace $\mathcal{A}_1 \supset \mathcal{A}_7 : (\theta_1, \ldots, \theta_6) = (a, b, a, a+\pi, b, a+\pi)$.

Weak chimera in reduced system of six phase oscillators



Phase portraits for the reduced system in variables $\xi = \theta_i - \theta_j \in [0, 2\pi)$, $\eta = \theta_i - \theta_k \in [0, 2\pi)$ for different values of parameter. The periodic, homoclinic and heteroclinic orbits that wind around the ξ direction of the torus $\mathbb{T}^2$ are *weak chimera states*.



Schematic representation of "serpent chimera" in variables $(\psi_1, \psi_2, \psi_3) \in \mathbb{T}^3$ for the two-modular system of two-component blocks. **(a)** The structure of the boundary surfaces (grey) for periodic *chimera states* (phase unlocked along ψ_2 periodic solutions in $\mathbb{T}^3$). **(b)** Projection into (ψ_1, ψ_3) plane. **(c)** "skin of chimera-snake" – dynamics on the surface of boundary chimera surface (the map from the surface to a plane).

Summary

- Phase oscillator networks are a simple and convenient way to describe the nature of synchronization and collective dynamics of many natural processes
- Phase models reflect many collective processes that are also inherent in more complex systems
- Phase models are easily upgraded to describe specific physical or biological processes
- Kuramoto model is the most popular and convenient of the phase models (Winfree model is the second)
- Some collective regimes and new mathematical theories (Watanabe-Strogats, Ott-Antonsen, etc.) were discovered precisely because of the phase models
- Network symmetries play the important role in the formation of collective regimes (clusters, splay states, slow switching, etc.) and invariant manifolds of the system
- Bifurcation theory is a very important tool in the study of collective dynamics
- New types of bifurcations can be identified and described in the study of networks of coupled elements