CAUCHY PROBLEM OF THE FRACTIONAL ORDER LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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1. Introduction

In work [6], the solution of the fractional order linear differential equations with constant coefficients (SFOLDECC) based on the Mittag-Leffler function is considered. However, in [3, 4] such solution is given for the first time on the basis of an exponential function, which from a computational point of view is more appropriate, since the exponential function can be calculated quite accurately [2, 5]. At the same time, in addition to the exponential function, this solution also includes an integral expression, which presents certain difficulties for the calculation [1]. Therefore, if instead of the exponential function we substitute its Taylor expansion, then we can obtain these integrals analytically, but the errors of the solution depend on the selected number of terms from the reduced series [9].

In this paper, we obtain analytical solution formulas, which, in contrast to [4], contain only one integral expression, which also makes it easier to obtain the solution of SFOLDECC. Next, substituting the expansion of the exponential function in the integrand, we obtain the numerical-analytical formula for the solution in the form of a Taylor series. The results are illustrated with a numerical example and a comparative analysis with the results of [6, 8] is given.

2. Simplified formula of solution the Cauchy problem

As is known for solving of Cauchy problem SFOLDECC

$$D^{\alpha}x\left(t\right)=Ax\left(t\right),\quad x\left(t_0\right)=x_0,\quad t>t_0>0, \tag{1}$$

there is the following analytical formula [4]

$$\begin{aligned} x\left(t\right)=&\left\{\sum_{s=0}^{2q}A^{\frac{s+2q+1}{2p+1}}\left[E\frac{t_0^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!}+A^{\frac{2q+1}{2p+1}}\int\limits_0^{t_0}\frac{\left(t_0-\tau\right)^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!}e^{\tau A^{\frac{2q+1}{2p+1}}}d\tau+\sum_{s=0}^{2q}A^{\frac{s}{2p+1}}\frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right.\right\}^{-1}\\ &\times\left\{\sum_{s=0}^{2q}A^{\frac{s+2q+1}{2p+1}}\left[E\frac{t^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!}+A^{\frac{2q+1}{2p+1}}\int\limits_0^{t}\frac{\left(t-\tau\right)^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!}e^{\tau A^{\frac{2q+1}{2p+1}}}d\tau\right]+\sum_{s=0}^{2q}A^{\frac{s}{2p+1}}\frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right\}x\left(t_0\right), \end{aligned} \tag{2}$$

where A is the constant matrix of dimension $n\times n$, $x\left(t\right)$is the states vector of dimension n, $x\left(t_0\right)$ is the initial vector, $\alpha\in\left(0,\ 1\right)$ and $\alpha=\frac{2p+1}{2q+1}$, here p and q are natural numbers, E is an unit matrix of dimension $n\times n$.

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As shown in [3], any real number can be approximated with any accuracy to rational numbers, and any rational number to numbers equal to the ratio of two odd numbers.

To simplify formulas (2), we use the relation [10]

$$D_{x_0}^{\alpha}u(\xi)=\frac{u(x_0)\,\xi^{-\alpha}}{(-\alpha)!}+\int_{x_0}^{\xi}\frac{(\xi-t)^{-\alpha}}{(-\alpha)!}u'(t)\,dt, \tag{3}$$

which defines derivatives $u(x)$ with order α. Using the expression (3) in (2), after simple transformations [3, 4] we reduce the relation (2) to the form

$$x(t)=\left[\sum_{s=0}^{2q}A^{\frac{s}{2p+1}}\frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right]^{-1}\sum_{s=0}^{2q}\left[A^{\frac{s}{2p+1}}\frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}+A^{\frac{s+2q+1}{2p+1}}\int_{t_0}^{t}\frac{(t-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}e^{A^{\frac{2q+1}{2p+1}}(\tau-t_0)}d\tau\right]x_0. \tag{4}$$

Note that in formula (4) the integrand function contains a weak singularity, since $\frac{s-2q}{2q+1}<1$, and this allows the disappearance of such singularity after integration.

Thus, the following statement holds:

Theorem 1. Let there be a Cauchy problem (1), where the coefficients A are constant matrices. Then its solution is represented in the form (4). Note that formula (4) in the classical case, i.e. when $\alpha=1$ coincides with the known [5]. Indeed in this case a$p=0$, $q=0$ and

$$x(t)=e^{-At_0}\left[e^{At_0}+A\int_{t_0}^{t}e^{A\tau}d\tau\right]x_0=\left[E+e^{-At_0}\left(e^{At}-e^{At_0}\right)\right]x_0=e^{A(t-t_0)}x_0. \tag{5}$$

3. Solution of the Cauchy problem (1) with constant perturbations

Let consider the following Cauchy problem

$$D^{\alpha}x(t)=Ax(t)+B(t)\ ,\quad x(t_0)=x_0, \tag{6}$$

where $B(t)$ is the vector of dimension n.

In the case when the vector $B(t)=B$ is constant, it is easy to show that in (12) the integral is calculated and we have the following analytical formula

$$x(t)=\sum_{k=0}^{\infty}A^k\frac{(t-t_0)^{(k+1)\alpha-1}}{[(k+1)\alpha-1]!}x_0+\sum_{k=0}^{\infty}A^k\frac{(t-t_0)^{(k+1)\alpha}}{[(k+1)\alpha]!}B. \tag{7}$$

Function (7) is a solution of the Cauchy problem (6) using the Mittag-Lefler function. Now, using expressions (4), we transform function (7) through the exponential function in the following form [7]

$$x(t)=\left[\sum_{s=0}^{2q}A^{\frac{s}{2p+1}}\frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}\right]^{-1}\left\{\sum_{s=0}^{2q}\left[A^{\frac{s}{2p+1}}\frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}+A^{\frac{s+2q+1}{2p+1}}\int_{t_0}^{t}\frac{(t-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}e^{(\tau-t_0)\cdot A^{\frac{2q+1}{2p+1}}}d\tau\right]x_0+\right.$$

$$\left.+\left[\int_{t_0}^{t}\sum_{s=0}^{2q}\left(A^{\frac{s}{2p+1}}\frac{\eta^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}+A^{\frac{s+2q+1}{2p+1}}\int_{t_0}^{\eta}\frac{(\eta-\tau)^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!}e^{(\tau-t_0)A^{\frac{2q+1}{2p+1}}}d\tau\right)d\eta\right]B\right\}. \tag{8}$$

Thus, the following theorem is proved

Theorem 2. Let in the Cauchy problem (6) $B(t)=B$ is the constant matrix. Then the solution of the corresponding Cauchy problem is represented in the form (8).

Note that at $p=0,\ q=0$ (i.e. $\alpha=1$) from (8) we have

$$\begin{aligned} x(t) &= \left\{ \left[E + A\int\limits_{t_0}^{t} e^{(\tau-t_0)A} d\tau \right] x_0 + \int\limits_{t_0}^{t} \left[E + A\int\limits_{t_0}^{\eta} e^{(\tau-t_0)A} d\tau \right] d\eta \cdot B \right\} \\ &= e^{A(t-t_0)}x_0 + \int\limits_{t_0}^{t} e^{A(\eta-t_0)} d\eta \cdot B = e^{A(t-t_0)}x_0 + A^{-1}\left(e^{A(t-t_0)} - E \right) \cdot B, \end{aligned} \tag{9}$$

which coincides with the classical solution (at $\alpha=1$) from [5].

Taking into account (4) in (8), and after some transformation we obtain the following expression for calculating of the solution of the Cauchy problem (6)

$$\begin{aligned} x(t) = &\left(\sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \frac{t_0^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right)^{-1} \left\{ \sum_{s=0}^{2q} \left(A^{\frac{s}{2p+1}} \frac{t^{\frac{s-2q}{2q+1}}}{\frac{s-2q}{2q+1}!} \right.\right. \\ &\left. - \frac{e^{(t-t_0)A^{\frac{2q+1}{2p+1}}}}{\frac{s-2q}{2q+1}!} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} A^{\frac{s+(k+1)(2q+1)}{2p+1}} \cdot \frac{(t-t_0)^{k+\frac{s+1}{2q+1}}}{k+\frac{s+1}{2q+1}} \right) x_0 \\ &+ \sum_{s=0}^{2q} A^{\frac{s}{2p+1}} \left[\left(\sum_{k=0}^{\infty} (-1)^k \frac{A^{(k+1)(2q+1)+s}}{\frac{s-2q}{2q+1}! \cdot k! \cdot \left(k+\frac{s+1}{2q+1}\right)} \right.\right. \\ &\left.\left.\left. \times \sum_{l=0}^{\infty} \frac{A^{l\frac{2q+1}{2p+1}}}{l!} \frac{(t-t_0)^{l+k+\frac{s+1}{2q+1}+1}}{\left(l+k+1+\frac{s+1}{2q+1}\right)} \right) + \left(\frac{t^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} - \frac{t_0^{\frac{s+1}{2q+1}}}{\frac{s+1}{2q+1}!} \right) \right] B \right\}. \end{aligned} \tag{10}$$

Keywords: Cauchy problem, linear fractional derivative system, Mittag-Leffler function, constant matrix coefficients.

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