Chaos, Solitons and Fractals 143 (2021) 110517

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

A singular boundary value problem for evolution equations of hyperbolic type

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ARTICLE INFO

Article history:
Received 28 June 2020
Revised 21 October 2020
Accepted 23 November 2020

MSC:
34B16
35A09
35L51
35L67
35L81

Keywords:
System of evolution equations of hyperbolic type
Bounded in a strip solution
Singular boundary value problem
Family of ordinary differential equations
Non-uniform partition
Method of parametrization
Solvability

ABSTRACT

This paper deals with a problem of finding a bounded in a strip solution to a system of second order hyperbolic evolution equations, where the matrix coefficient of the spatial derivative tends to zero as $t \to \mp\infty$. The problem is studied under assumption that the coefficients, the right-hand side of the system, and the boundary function belong to some spaces of functions continuous and bounded with a weight. By introducing new unknown functions, the problem in question is reduced to an equivalent problem consisting of singular boundary value problems for a family of first order ordinary differential equations and some integral relations. Existence conditions are established for a bounded in a strip solutions to a family of ordinary differential equations, whose matrix tends to zero as $t \to \mp\infty$ and the right-hand side is bounded with a weight. Conditions for the existence of a unique solution to the original problem are obtained.

1. Introduction

We consider the system of hyperbolic equations with mixed derivatives

$$u_{xt} = A(x,t)u_x + B(x,t)u_t + C(x,t)u + f(x,t), \quad (1)$$

in a strip $\Omega^* = [0,\omega] \times (-\infty,\infty)$. Here $u_x(x,t) = \frac{\partial u(x,t)}{\partial x}$, $u_t(x,t) = \frac{\partial u(x,t)}{\partial t}$, $u_{xt}(x,t) = \frac{\partial^2 u(x,t)}{\partial t \partial x}$.

In recent years, various classes of hyperbolic evolution equations and singular problems for them have been extensively studied. It is associated primarily with their numerous applications in physics, biology, chemistry, etc. Evolution equations arise in mathematical modeling of various processes and phenomena in natural science, acoustics, and neural networks [1,4–8,21–23,27–30]. A number of significant results have been obtained for a wide class of evolution equations; see [5,23,29,30] and references therein. However, there has been little investigation of bounded solutions to systems of hyperbolic equations with mixed derivative (1). In [24–26], bounded in a strip solutions to systems of hyperbolic equations were studied in the case of diagonal dominance in the matrix $A(x,t)$; necessary conditions for their existence and necessary and sufficient conditions for their uniqueness were obtained.

We will use the following notation:

$C_*(\Omega^*,\mathbb{R}^n)$ is the space of bounded functions $u : \Omega^* \to \mathbb{R}^n$ that are continuous with respect to t and uniformly continuous in x with respect to t, with the norm $\|u\|_* = \sup_{(x,t)\in\Omega^*} \|u(x,t)\|$;

$C_*(\mathbb{R},\mathbb{R}^n)$ is the space of bounded functions $\psi : \mathbb{R} \to \mathbb{R}^n$ with the norm $\|\psi\|_* = \sup_{t\in\mathbb{R}} \|\psi(t)\|$.

Suppose the columns of the matrices $A(x,t)$, $B(x,t)$, $C(x,t)$, and the vector-function $f(x,t)$ belong to $C_*(\Omega^*,\mathbb{R}^n)$. We are interested

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[1] This research is funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08855726). The authors (with other colleagues) have received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 873071.

https://doi.org/10.1016/j.chaos.2020.110517
0960-0779/

in the solutions to system (1) that satisfy the conditions

$$u(0,t)=\psi(t), \quad t\in\mathbb{R}, \qquad u_x(x,t)\in C_*(\Omega^*,R^n), \tag{2}$$

where $\psi(t)\in C_*(\mathbb{R},\mathbb{R}^n)$ is a continuously differentiable function with $\dot\psi(t)\in C_*(\mathbb{R},\mathbb{R}^n)$.

A classical solution to problem (1), (2) is a function $u(x,t)$ that has continuous partial derivatives $u_x(x,t)$, $u_t(x,t)$, and $u_{xt}(x,t)$ in Ω^* and, for all $(x,t)\in\Omega^*$, satisfies system (1) and additional conditions (2).

It follows from (2) and the boundedness of $\psi(t)$ on $\mathbb{R}$ that $u(x,t)\in C_*(\Omega^*,\mathbb{R}^n)$. Since $\dot\psi(t)$ is bounded on $\mathbb{R}$, from (1) we deduce that $u_t(x,t)$ and $u_{xt}(x,t)$ belong to $C_*(\Omega^*,\mathbb{R}^n)$ as well.

In [2], problem (1), (2) was studied by the parametrization method [18]. Sufficient conditions for the unique solvability of the problem were obtained in terms of a two-sided infinite matrix $Q_{\nu,h}(x)$ constructed via $A(x,t)$. It was shown that the main condition for the unique solvability is the bounded invertibility of $Q_{\nu,h}(x)$. The properties of bounded solutions to problem (1), (2) were established and approximating boundary value problems in a finite region were constructed. In [3], the results obtained in [2] were applied to finding bounded periodic solutions to system (1).

Of particular interest are singular boundary value problems for evolution equations with coefficients or right-hand sides tending to zero as time variable approaches infinity. One of such problems is the problem of finding a bounded solution to a nonhomogeneous linear system of ordinary differential equations whose matrix tends to zero as $t\to\mp\infty$. It is known that in this case the corresponding homogeneous system does not admit an exponential dichotomy, i.e. not for every continuous and bounded right-hand side the nonhomogeneous system has a solution bounded on the whole real line [10]. The same issue occurs for singular boundary value problems for hyperbolic evolution Eq. (1) in the case when $A(x,t)$ approaches zero as $t\to\mp\infty$.

Suppose that the matrix $A(x,t)$ satisfies the condition $\|A(x,t)\|\equiv\max\limits_j\sum\limits_{k=1}^n|a_{jk}(x,t)|\le\alpha(t)$, where $\alpha(t)$ is a continuous and positive on $\mathbb{R}$ function such that

$$\int_{-\infty}^0\alpha(t)dt=\infty, \qquad \int_0^\infty\alpha(t)dt=\infty; \tag{3}$$

$$\lim_{t\to-\infty}\alpha(t)=0, \qquad \lim_{t\to+\infty}\alpha(t)=0. \tag{4}$$

It is known that under assumptions (3), (4) problem (1), (2) has a classical solution not for all $f(x,t)\in C_*(\Omega^*,\mathbb{R}^n)$. The following example illustrates this statement.

Example. Let us consider the following problem

$$u_{xt}=\frac{2t}{1+t^2}u_x+f(x,t), \qquad (x,t)\in\Omega^*, \tag{5}$$

$$u(0,t)=0, \qquad t\in\mathbb{R}, \qquad u_x(x,t)\in C_*(\Omega^*,\mathbb{R}^n). \tag{6}$$

Obviously, the function $\alpha(t)=\dfrac{2|t|+1}{1+t^2}$ satisfies (3), (4).

For $f(x,t)=0$, the set of solutions to (5) is $u(x,t)=(1+t^2)\int\limits_0^x C(\xi)d\xi$, where $C(x)$ is a continuous on $[0,\omega]$ function. The only classical solution to problem (5), (6) in this set is $u(x,t)=0$.

If $f(x,t)=1$, then $u(x,t)=(1+t^2)\int\limits_0^x C(\xi)d\xi+(1+t^2)x\,\mathrm{arctg}\,t$. None of these functions is a classical solution to problem (5), (6).

The question now arises: is it possible to solve the problem of finding a classical solution to problem (1), (2) under assumptions (3), (4), if we place some additional requirements on the input data of problem?

In this paper, we study the existence and uniqueness of a classical solution to problem (1), (2) subject to some extra assumptions regarding the coefficients $B(x,t)$ and $C(x,t)$, the right-hand side $f(x,t)$, and the derivative of the boundary function $\psi(t)$.

To this end, we introduce the following spaces of functions continuous and bounded with a weight on Ω^* and $\mathbb{R}$ (a weight function is chosen so that the behavior of $A(x,t)$ as $t\to\mp\infty$ is taken into account):

$C_{*,1/\alpha}(\Omega^*,\mathbb{R}^n)$ is the space of functions $f:\Omega^*\to\mathbb{R}^n$ that are bounded with the weight $1/\alpha(t)$, continuous in $t\in\mathbb{R}$ for $x\in[0,\omega]$, and uniformly continuous in $x\in[0,\omega]$ for $t\in\mathbb{R}$, with the norm $\|f\|_\alpha=\sup\limits_{(x,t)\in\Omega^*}\|f(x,t)/\alpha(t)\|$;

$C_{*,1/\alpha}(\mathbb{R},\mathbb{R}^n)$ is the space of bounded with the weight $1/\alpha(t)$ functions $\psi:\mathbb{R}\to\mathbb{R}^n$ with the norm $\|\psi\|_\alpha=\sup\limits_{t\in\mathbb{R}}\|\psi(t)/\alpha(t)\|$.

Under assumptions (3),(4), we pose **singular boundary value problem (1), (2)**: find a classical solution to problem (1),(2) if the columns of the matrix $C(x,t)$ and function $f(x,t)$ belong to $C_{*,1/\alpha}(\Omega^*,\mathbb{R}^n)$ and $\dot\psi(t)\in C_{*,1/\alpha}(\mathbb{R},\mathbb{R}^n)$.

The rest of the paper is organized as follows. In Section 2 problem (1), (2) is reduced to an equivalent problem consisting of singular boundary value problems for a family of first order ordinary differential equations and integral relations. This problem can be interpreted as the problem of finding a bounded in a strip solution to a family of systems of ordinary differential equations with two unknown functions that are connected with the desired solution and its derivative via some integral relations. This requires a separate study.

Section 3 is therefore devoted to the problem of finding a bounded in a strip solution to a family of systems of ordinary differential equations with the matrix tending to zero as $t\to\mp\infty$ and the right-hand side bounded with a weight. The strip Ω^* is partitioned taking into account the behavior of a function that is an upper bound of $\|A(x,t)\|$. The problem in question is thereby transformed into an equivalent problem with parameters that are equal to the values of the solution at the partition points. Since the solution to original problem is bounded, the sequence of parameters is required to be bounded as well. We have obtained conditions for the unique solvability of the problem in terms of a two-sided infinite block band matrix that is composed of sums of iterated integrals of $A(x,t)$ over the partition subintervals.

In Section 4, based on the results of the previous section, we have established conditions for the existence of a unique solution to singular boundary value problem (1), (2).

2. Reduction problem (1), (2) to a boundary value problem for a family of ordinary differential equations and integral relations

By introducing new unknown functions $v(x,t)=u_x(x,t)$ and $w(x,t)=u_t(x,t)$, problem (1), (2) is reduced to an equivalent problem

$$v_t=A(x,t)v+F(x,t,w(x,t),u(x,t)),\ v(x,t)\in C_*(\Omega^*,R^n), \tag{2.1}$$

$$u(x,t)=\psi(t)+\int_0^x v(\xi,t)d\xi,$$
$$w(x,t)=\dot\psi(t)+\int_0^x v_t(\xi,t)d\xi, \tag{2.2}$$

where $(x,t)\in\Omega^*$, $F=B(x,t)w(x,t)+C(x,t)u(x,t)+f(x,t)$. The condition $u(0,t)=\psi(t)$ is taken into account in relations (2.2).

A triple of functions $\{v(x,t),u(x,t),w(x,t)\}$ continuous on Ω^* is called a solution to problem (2.1), (2.2) if the function $v(x,t)\in C_*(\Omega^*,R^n)$ is continuously differentiable with respect to t in Ω^*

and satisfies the one-parameter family of systems of ordinary differential Eq. (2.1), and the functions $u(x,t)$ and $w(x,t)$ are connected with $v(x,t)$ and $v_t(x,t)$ by the integral relations (2.2).

Problems (1), (2) and (2.1), (2.2) are equivalent in the following sense. Let $u(x,t)$ be a classical solution to problem (1), (2). Then, if we compose the triple of functions $\{v(x,t), u(x,t), w(x,t)\}$ with $v(x,t) = u_x(x,t)$ and $w(x,t) = u_t(x,t)$, we have

$$u(x,t) = u(0,t) + \int_0^x u_\xi(\xi,t)d\xi = \psi(t) + \int_0^x v(\xi,t)d\xi,$$

$$w(x,t) = u_t(x,t) = u_t(0,t) + \int_0^x u_{t\xi}(\xi,t)d\xi = u_t(0,t) + \int_0^x u_{\xi t}(\xi,t)d\xi = \dot\psi(t) + \int_0^x v_t(\xi,t)d\xi,$$

$$\begin{aligned} v_t &= u_{xt} = A(x,t)u_x + B(x,t)u_t + C(x,t)u + f(x,t) \\ &= A(x,t)v + F(x,t,w(x,t),u(x,t)), \end{aligned}$$

$$v(x,t) = u_x(x,t) \in C_*(\Omega^*, R^n).$$

Hence the triple composed is a solution to problem (2.1), (2.2).

Conversely, if a function triple $\{v(x,t), u(x,t), w(x,t)\}$ is a solution to problem (2.1), (2.2), then it follows from (2.2) that $u(x,t)$ satisfies the condition $u(0,t) = \psi(t)$ and has continuous partial derivatives $u_x(x,t) = v(x,t) \in C_*(\Omega^*, R^n)$, $u_t(x,t) = \dot\psi(t) + \int\limits_0^x v_t(\xi,t)d\xi = w(x,t)$, $u_{xt}(x,t) = v_t(x,t)$, $u_{tx}(x,t) = v_t(x,t)$. Substituting them into the right-hand side of (2.1) we get $u(x,t)$ satisfies (1) for all $(x,t) \in \Omega^*$. Since this function also satisfies (2), we conclude that $u(x,t)$ is a classical solution to problem (1), (2).

For fixed $u(x,t)$, $w(x,t)$ in problem (2.1), (2.2), we need to find a solution to the one-parameter family of ordinary differential equations that belongs to $C_*(\Omega^*, R^n)$. Therefore, in conjunction with problem (2.1), (2.2), we study the following problem.

In Ω^*, we consider the family of ordinary differential equations

$$v_t = A(x,t)v + F(x,t), \qquad v(x,t) \in C_*(\Omega^*, R^n), \tag{2.3}$$

under assumption that the columns of the matrix $A(x,t)$ and the vector-function $F(x,t)$ belong to $C_*(\Omega^*, R^n)$.

The problem of finding a function $v(x,t) \in C_*(\Omega^*, R^n)$ that satisfies (2.3) for all $(x,t) \in \Omega^*$ will be referred as **Problem 1**.

Problem 1 was studied in [13,14] by the parametrization method with uniform partitioning. Necessary and sufficient conditions for the unique solvability were obtained in terms of a two-sided infinite block band matrix that is constructed via integrals of $A(x,t)$ over intervals of length $h > 0$. In [15,16], based on the equivalence of Problem 1 and problem (1), (2), a coefficient criterion for the well-posedness of the latter problem was established.

For fixed $x \in [0,\omega]$ Problem 1 becomes the problem of finding a solution to the system of ordinary differential equations, that is bounded on the whole real line. This problem was studied in [9–12]; in particular, the questions of solvability, exponential dichotomy of the system, as well as approximation by solutions of boundary value problems on finite intervals were considered. A number of fundamental results were obtained on the basis of the parametrization method.

The parametrization method was originally developed to study regular two-point boundary value problems; in [18] necessary and sufficient conditions for their unique solvability were established. It was shown that one of the main conditions is the invertibility of a matrix $Q_\nu(h)$, composed of the matrices of boundary conditions and the system of differential equations for some $\nu (\nu = 1, 2, \ldots)$ and $h > 0$. In [9], the parametrization method was applied to the problem of finding a bounded on $\mathbb{R}$ solution to a linear ordinary differential equation. Necessary and sufficient conditions for the well-posedness of singular boundary value problems were established in terms of a two-sided infinite block band matrix $Q_{\nu,h}$, whose entries are composed through the sums of iterated integrals of the matrix of the equation over the partition intervals. The formulation of conditions for the well-posedness of regular and singular boundary value problems in unified terms of matrices $Q_\nu(h)$ and $Q_{\nu,h}$, respectively, made it possible to solve the problem of approximation a singular problem by two-point boundary value problems on a finite interval. The results obtained were extended to families of systems of ordinary differential equations [17] and nonlinear problems [19].

Similarly to the above-mentioned problems, under assumptions (3), (4) Problem 1 admits a bounded in Ω^* solution not for every $F(x,t) \in C_*(\Omega^*, R^n)$. Indeed, let us consider the following problem that is analogous to problem (5), (6):

$$v_t = \frac{2t}{1+t^2}v + F(x,t), \qquad v(x,t) \in C_*(\Omega^*, \mathbb{R}^n).$$

If $F(x,t) = 0$ then the set of all solutions to the differential equation is of the form $v(x,t) = C(x)(1+t^2)$, where $C(x)$ is a function continuous on $[0,\omega]$. In this set, $v(x,t) = 0$ is the only solution bounded in Ω^*.

But, if we take $F(x,t) = 1$, the set of all solutions of the equation is of the form $v(x,t) = (1+t^2)(C(x) + \mathrm{arctg}\, t)$. None of this solutions are bounded in Ω^*.

The same question arises as to whether it is possible to solve the problem of finding a bounded in Ω^* solution to Eq. (2.3) under certain requirements to the right-hand side $F(x,t)$. In the next section we study the singular boundary value problem assuming that $F(x,t) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$.

3. Singular boundary value problems for a family of systems of ordinary differential equations

In Ω^*, consider the family of systems of differential equations

$$v_t = A(x,t)v + F(x,t), \tag{3.1}$$

where the columns of $A(x,t)$ and $F(x,t)$ belong to the space $C_*(\Omega^*, \mathbb{R}^n)$. We assume $\|A(x,t)\| \le \alpha(t)$, where $\alpha(t) > 0$ is a function continuous on $\mathbb{R}$ and satisfying conditions (3), (4).

The problem of finding a bounded in Ω^* solution to (3.1), when $F(x,t) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$, will be referred to as **Problem** 1_α.

The problem of finding a solution to a system of ordinary differential equations, that is bounded with a weight on the whole real line, was studied in [20,31–33] by the parametrization method with nonuniform partitioning. Necessary and sufficient conditions were obtained for the well-posedness of the problem of finding a bounded on $\mathbb{R}$ solution to a linear ordinary differential equation with the matrix tending to zero as $t \to \mp\infty$ and the right-hand side bounded with a weight. Regular two-point boundary value problems were constructed, that enable one to find an approximate solution to the singular problem to within a given accuracy. The mutual relationship between the solvability of the singular problem and that of approximating regular problems was established.

We now proceed to the study of Problem 1_α using the parametrization method with a nonuniform partition of the strip Ω^*.

We choose a number $\theta > 0$ and make the partition $\Omega^* = \bigcup\limits_{s=-\infty}^{\infty} [0,\omega] \times [t_{s-1}, t_s)$, where the points $t_s \in \mathbb{R}$, $s \in \mathbb{Z}$, are defined as follows: $t_0 = 0$, $\int\limits_{t_{s-1}}^{t_s} \alpha(t)dt = \theta$.

Let $\widetilde{h}(\theta)$ denote the two-sided infinite sequence of the lengths of the partition subintervals $h_s = t_s - t_{s-1}$, $s \in \mathbb{Z}$, i.e., $\widetilde{h}(\theta) = (\ldots, h_s(\theta), h_{s+1}(\theta), \ldots)'$.

We introduce the following notation:

m_n is the space of bounded two-sided infinite sequences $\lambda = (\ldots, \lambda_s, \lambda_{s+1}, \ldots)'$ of vectors $\lambda_s \in \mathbb{R}^n$, $s \in \mathbb{Z}$, with the norm $\|\lambda\|_{m_n} = \sup_s \|\lambda_s\|$;

$m_n(\widetilde{h}(\theta))$ is the space of bounded two-sided infinite sequences $v[t] = (\ldots, v_s(t), v_{s+1}(t), \ldots)'$ of continuous functions $v_s : [t_{s-1}, t_s) \to R^n$, $s \in \mathbb{Z}$, with the norm $\|v[\cdot]\|_{m_n(\tilde{h}(\theta))} = \sup_s \sup_{t \in [t_{s-1}, t_s)} ||v_s(t)||$;

$C([0,\omega], m_n)$ is the space of two-sided infinite sequences of continuous on $[0,\omega]$ functions $\lambda(x) = (\ldots, \lambda_s(x), \lambda_{s+1}(x), \ldots)'$ with the norm $\|\lambda\|_1 = \max_{x \in [0,\omega]} \|\lambda(x)\|_{m_n}$;

$C([0,\omega], m_n(\widetilde{h}(\theta)))$ is the space of continuous mappings $v : [0,\omega] \to m_n(\widetilde{h}(\theta))$, $v(x,[t]) = (\ldots, v_s(x,t), v_{s+1}(x,t), \ldots)'$, with the norm $\|v\|_2 = \max_{x \in [0,\omega]} \|v(x,[\cdot])\|_{m_n(\tilde{h}(\theta))}$;

$L(X)$ is the space of bounded linear operators $Z : X \to X$ with the induced norm, where X is a Banach space.

Let $v_s(x,t)$ be the restriction of the function $v(x,t) \in C_*(\Omega^*, \mathbb{R}^n)$ onto the subregion $\Omega_s = [0,\omega] \times [t_{s-1}, t_s)$, $s \in \mathbb{Z}$; i.e., $v_s(x,t) = v(x,t)$ for $(t,x) \in \Omega_s$. On each Ω_s, $s \in \mathbb{Z}$, we make the substitution $\widetilde{v}_s(x,t) = v_s(x,t) - \lambda_s(x)$, where $\lambda_s(x) = v_s(x, t_{s-1})$. The Problem 1_α is then transformed into the equivalent family of multipoint boundary value problems with parameters

$$\frac{\partial \tilde{v}_s}{\partial t} = A(x,t)[v_s + \lambda_s(x)] + F(x,t), \ (x,t) \in \Omega_s, \tag{3.2}$$

$$\widetilde{v}_s(x, t_{s-1}) = 0, \qquad x \in [0,\omega], \quad s \in \mathbb{Z}, \tag{3.3}$$

$$\lim_{t \to t_s - 0} \tilde{v}_s(x,t) + \lambda_s(x) = \lambda_{s+1}(x), x \in [0,\omega], s \in \mathbb{Z}, \tag{3.4}$$

$$(\lambda(x), \widetilde{v}(x,[t])) \in C([0,\omega], m_n) \times C([0,\omega], m_n(\widetilde{h}(\theta))). \tag{3.5}$$

Problem (3.2)–(3.5) and Problem 1_α are equivalent in the following sense. If a pair $(\lambda^*(x), \widetilde{v}^*(x,[t]))$ is a solution to problem (3.2)–(3.5), then the function $v^*(x,t)$, defined as

$$v^*(x,t) = \lambda_s^*(x) + \tilde{v}_s^*(x,t), \ (x,t) \in \Omega_s, s \in \mathbb{Z},$$

belongs to $C_*(\Omega^*, \mathbb{R}^n)$ and satisfies (3.1) for all $(x,t) \in \Omega^*$, i.e. $v^*(x,t)$ is a solution to Problem 1. Furthermore, if we assume $F(x,t) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$, then $v^*(x,t)$ is a solution to Problem 1_α. Vice versa, if $v(x,t)$ is a solution to Problem 1_α, i.e. it is a solution to Problem 1 when $F(x,t) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$, then the pair $(\lambda(x), \widetilde{v}(x,[t]))$, where

$$\lambda(x) = (\ldots, v_s(x, t_{s-1}), v_{s+1}(x, t_s), \ldots)',$$

$$\widetilde{v}(x,[t]) = (\ldots, v_s(x,t) - v_s(x,t_{s-1}), v_{s+1}(x,t) - v_{s+1}(x,t_s), \ldots)',$$

satisfies the family of differential equations (3.2), initial conditions (3.3), continuity conditions at the partition points (3.4), and boundedness condition (3.5).

For fixed values of the parameter $\lambda_s(x)$, the family of Cauchy problems (3.2), (3.3) has a unique solution $\widetilde{v}_s(x,t)$ satisfying the family of integral equations

$$\tilde{v}_s(x,t) = \int_{t_{s-1}}^{t} A(x,\tau)[\tilde{v}_s(x,\tau) + \lambda_s(x)]d\tau + \int_{t_{s-1}}^{t} f(x,\tau)d\tau, \ (x,t) \in \Omega_s, s \in \mathbb{Z}. \tag{3.6}$$

Replacing $\widetilde{v}_s(x,\tau)$ by the corresponding right-hand side of the Eq. (3.6) and repeating this procedure $\nu (\nu = 1, 2, \ldots)$ times, we get

$$\widetilde{v}_s(x,t) = D_{\nu,s}(h_s(\theta), x)\lambda_s(x) + F_{\nu,s}(h_s(\theta), x) + G_{\nu,s}(\widetilde{v}, h_s(\theta), x), \tag{3.7}$$

where

$$D_{\nu,s}(h_s(\theta), x) = \int_{t_{s-1}}^{t_s} A(x,\tau_1)d\tau_1 + \int_{t_{s-1}}^{t_s} A(x,\tau_1)\int_{t_{s-1}}^{\tau_1} A(x,\tau_2)d\tau_2 d\tau_1 +$$

$$+ \ldots + \int_{t_{s-1}}^{t_s} A(x,\tau_1)\ldots\int_{t_{s-1}}^{\tau_{\nu-1}} A(x,\tau_\nu)d\tau_\nu \ldots d\tau_1,$$

$$F_{\nu,s}(h_s(\theta), x) = \int_{t_{s-1}}^{t_s} F(x,\tau_1)d\tau_1 + \int_{t_{s-1}}^{t_s} A(x,\tau_1)\int_{t_{s-1}}^{\tau_1} F(x,\tau_2)d\tau_2 d\tau_1 + \ldots +$$

$$+ \int_{t_{s-1}}^{t_s} A(x,\tau_1)\ldots\int_{t_{s-1}}^{\tau_{\nu-2}} A(x,\tau_{\nu-1})\int_{t_{s-1}}^{\tau_{\nu-1}} F(x,\tau_\nu)d\tau_\nu d\tau_{\nu-1}\ldots d\tau_1,$$

$$G_{\nu,s}(\tilde{v}, h_s(\theta), x) = \int_{t_{s-1}}^{t_s} A(x,\tau_1)\ldots\int_{t_{s-1}}^{\tau_{\nu-1}} A(x,\tau_\nu)u_s(\tau_\nu)d\tau_\nu \ldots d\tau_1.$$

From (3.4), substituting $\lim_{t\to t_s-0}\widetilde{v}_s(x,t)$, $s\in\mathbb{Z}$, by their expressions derived from (3.7), we get the two-sided system of functional equations in parameters $\lambda_s(x)$:

$$[I+D_{\nu,s}(h_s(\theta),x)]\lambda_s(x)-\lambda_{s+1}(x)=-F_{\nu,s}(h_s(\theta),x)-G_{\nu,s}(\tilde{v},h_s(\theta),x),x\in[0,\omega],s\in\mathbb{Z}, \tag{3.8}$$

where I is the identity matrix of order n.

Let $Q_\nu(\widetilde{h}(\theta),x)$ denote the block band matrix corresponding to the left-hand side of system (3.8). In each row of $Q_\nu(\widetilde{h}(\theta),x)$, the only nonzero elements are $I+D_{\nu,s}(h_s(\theta),x)$ and $-I$. Therefore, for all $x\in[0,\omega]$, this matrix maps the space m_n into itself, and $||Q_\nu(\widetilde{h}(\theta),x)||_{L(m_n)}\le 2+\sum_{j=1}^{\nu}\frac{\theta^j}{j!}$. Since the columns of the matrix $A(x,t)$ belong to $C_*(\Omega^*,\mathbb{R}^n)$, the matrix $Q_\nu(\widetilde{h}(\theta),x)$ is continuous with respect to $x\in[0,\omega]$ in the norm of $L(m_n)$ and $Q_\nu(\widetilde{h}(\theta),x)\in L(C([0,\omega],m_n))$ for all $\nu\in\mathbb{N}$.

We rewrite system (3.8) in the form

$$Q_\nu\left(\tilde{h}(\theta),x\right)\lambda(x)=-F_\nu\left(\tilde{h}(\theta),x\right)-G_\nu\left(\tilde{v},\tilde{h}(\theta),x\right),\lambda(x)\in C([0,\omega],m_n), \tag{3.9}$$

where

$$F_\nu(\widetilde{h}(\theta),x)=(\dots,F_{\nu,s}(h_s(\theta),x),F_{\nu,s+1}(h_{s+1}(\theta),x),\dots)',$$

$$G_\nu(u,\widetilde{h}(\theta),x)=(\dots,G_{\nu,s}(u,h_s(\theta),x),G_{\nu,s+1}(u,h_{s+1}(\theta),x),\dots)'.$$

It follows from the inequalities

$$||F_{\nu,s}(h_s(\theta),x)||\le\sum_{j=1}^{\nu}\frac{\theta^j}{j!}\|F\|_\alpha,\qquad \|G_\nu(\tilde{v},h_s(\theta),x)\|\le\frac{\theta^\nu}{\nu!}\|\tilde{v}(x,[t])\|_2,\qquad s\in\mathbb{Z},$$

that $F_\nu(\widetilde{h}(\theta),x)$ and $G_\nu(\widetilde{v},\widetilde{h}(\theta),x)$ belong to $C([0,\omega],m_n)$ for all $\theta>0$ and $\widetilde{v}(x,[t])\in C([0,\omega],m_n(\widetilde{h}(\theta)))$.

The solution to multipoint boundary value problem (3.2)–(3.5), the pair $(\lambda^*(x),\widetilde{v}^*(x,[t]))$, can be found as the limit of the sequence $(\lambda^{(k)}(x),\widetilde{v}^{(k)}(x,[t]))$, which is defined by the following algorithm.

Step 0. Assuming that for some chosen $\theta>0$ and $\nu\in\mathbb{N}$ the operator $Q_\nu(\widetilde{h}(\theta),x):C([0,\omega],m_n)\to C([0,\omega],m_n)$ is boundedly invertible, we determine an initial approximation of the functional parameter $\lambda^{(0)}(x)\in C([0,\omega],m_n)$ from the equation

$$Q_\nu(\widetilde{h}(\theta),x)\lambda(x)=-F_\nu(\widetilde{h}(\theta),x).$$

Then, solving the Cauchy problems (3.2), (3.3) on Ω_s, $s\in\mathbb{Z}$, for $\lambda_s(x)=\lambda_s^{(0)}(x)$, we find $\tilde{v}^{(0)}(x,[t])\in C([0,\omega],m_n(\tilde{h}(\theta)))$.

Step 1. Substituting $\widetilde{v}_s^{(0)}(x,t)$, $s\in\mathbb{Z}$, into the right-hand side of (3.9), we get the equation

$$Q_\nu(\widetilde{h}(\theta),x)\lambda(x)=-F_\nu(\widetilde{h}(\theta),x)-G_\nu(\widetilde{v}^{(0)},\widetilde{h}(\theta),x),$$

from which we find $\lambda^{(1)}(x)\in C([0,\omega],m_n)$. Solving the families of Cauchy problems (3.2), (3.3) on Ω_s, $s\in\mathbb{Z}$, for $\lambda_s(x)=\lambda_s^{(1)}(x)$, we get $\tilde{v}^{(1)}(x,[t])\in C([0,\omega],m_n(\tilde{h}(\theta)))$.

And so on.

The following theorem establishes convergence conditions of the algorithm proposed and provides an estimate of the solution to Problem 1_α.

Theorem 1. *Let, for some $\theta>0$ and $\nu(\nu=1,2,\dots)$, the functional matrix $Q_\nu(\widetilde{h}(\theta),x):m_n\to m_n$ be invertible for all $x\in[0,\omega]$ and the following inequalities hold:*

$$\|[Q_\nu(\widetilde{h}(\theta),x)]^{-1}\|_{L(m_n)}\le\gamma_\nu(\widetilde{h}(\theta)),\quad \gamma_\nu(h)-const; \tag{3.10}$$

$$q_\nu(\widetilde{h}(\theta))=\gamma_\nu(\widetilde{h}(\theta))\left[e^\theta-1-\theta-\dots-\frac{\theta^\nu}{\nu!}\right]<1. \tag{3.11}$$

Then Problem 1_α has a unique solution $v^*(x,t)$ and the estimate

$$\|v^*\|_*\le\left\{\gamma_\nu\left(\tilde{h}(\theta)\right)\left[\frac{e^\theta M\left(\tilde{h}(\theta)\right)}{1-q_\nu\left(\tilde{h}(\theta)\right)}\frac{\theta^\nu}{\nu!}+\sum_{j=1}^{\nu}\frac{\theta^j}{j!}\right]+M\left(\tilde{h}(\theta)\right)\right\}\|F\|_\alpha \tag{3.12}$$

is valid, where $M(\widetilde{h}(\theta))=\theta e^\theta+(e^\theta-1)\gamma_\nu(\widetilde{h}(\theta))\sum_{j=1}^{\nu}\frac{\theta^j}{j!}$.

Proof. It follows from the assumptions on the coefficients and the right-hand side of system (3.2) that $Q_\nu(\widetilde{h}(\theta),x):C([0,\omega],m_n)\to C([0,\omega],m_n)$. From (3.10) we conclude that $[Q_\nu(\widetilde{h}(\theta),x)]^{-1}\in L(C([0,\omega],m_n))$. Therefore, there exists a unique $\lambda^{(0)}(x)\in C([0,\omega],m_n)$, and

$$\|\lambda^{(0)}(x)\|_1\le\gamma_\nu\left(\tilde{h}(\theta)\right)\|F_\nu\left(\tilde{h}(\theta),x\right)\|_1\le\gamma_\nu\left(\tilde{h}(\theta)\right)\sum_{j=1}^{\nu}\frac{\theta^j}{j!}\|F\|_\alpha.$$

For $\lambda_s(x)=\lambda_s^{(0)}(x)$, the Cauchy problem (3.2), (3.3) has a unique solution $\widetilde{v}_s^{(0)}(x,t)$. By applying the Gronwall - Bellman inequality, we obtain

$$\|\tilde{v}_s^{(0)}(x,t)\|\le\theta e^\theta\|F\|_\alpha+\left(e^\theta-1\right)\|\lambda_s^{(0)}(x)\|,\qquad s\in\mathbb{Z},$$

$\|\tilde{v}^{(0)}\|_2 \leq M\big(\tilde{h}(\theta)\big)\|F\|_\alpha$.

Further, according to Algorithm, we find $\lambda^{(1)}(x)$ and obtain the estimate

$$\|\lambda^{(1)} - \lambda^{(0)}\|_1 \leq \gamma_\nu\big(\tilde{h}(\theta)\big)\|G_\nu\big(\tilde{v}^{(0)}, \tilde{h}(\theta), x\big)\|_1 \leq \gamma_\nu\big(\tilde{h}(\theta)\big)\frac{\theta^\nu}{\nu!}M\big(\tilde{h}(\theta)\big)\|F\|_\alpha. \tag{3.13}$$

Proceeding with the iteration process, we find the sequence of systems of pairs $(\lambda_s^{(k)}(x), \tilde{v}_s^{(k)}(x, [t]))$, $s \in \mathbb{Z}$, $k = 1, 2, \ldots$. Using again the Gronwall-Bellman inequality, we derive the estimate of the difference between the solutions of the Cauchy problems via the difference between the corresponding parameters:

$$\|\tilde{v}_s^k(x,t) - \tilde{v}_s^{(k-1)}(x,t)\| \leq \left(e^{\int_{t_{s-1}}^{t}\alpha(\tau)d\tau} - 1\right)\|\lambda_s^{(k)}(x) - \lambda_s^{(k-1)}(x)\|, \ (x,t) \in \Omega_s, \qquad s \in \mathbb{Z}. \tag{3.14}$$

From (3.9) and (3.14) we obtain the estimate

$$\|\lambda^{(k+1)} - \lambda^{(k)}\|_1 \leq q_\nu\big(\tilde{h}(\theta)\big)\|\lambda^{(k)} - \lambda^{(k-1)}\|_1, \qquad k = 1, 2, \ldots, . \tag{3.15}$$

It follows from condition (3.11) and inequalities (3.13)–(3.15) that the sequence $(\lambda^{(k)}(x), \tilde{v}^{(k)}(x,[t]))$ converges to $(\lambda^*(x), \tilde{v}^*(x,[t]))$ as $k \to \infty$ and the following estimates hold:

$$\|\lambda^* - \lambda^{(k)}(x)\|_1 \leq \frac{\big[q_\nu\big(\tilde{h}(\theta)\big)\big]^k}{1 - q_\nu\big(\tilde{h}(\theta)\big)}\gamma_\nu\big(\tilde{h}(\theta)\big)\frac{\theta^\nu}{\nu!}M\big(\tilde{h}(\theta)\big)\|F\|_\alpha,$$

$$\|\tilde{v}^* - \tilde{v}^{(k)}\|_2 \leq \big(e^\theta - 1\big)\frac{\big[q_\nu\big(\tilde{h}(\theta)\big)\big]^k}{1 - q_\nu\big(\tilde{h}(\theta)\big)}\gamma_\nu\big(\tilde{h}(\theta)\big)\frac{\theta^\nu}{\nu!}M\big(\tilde{h}(\theta)\big)\|F\|_\alpha.$$

Since $(\lambda^*(x), \tilde{v}^*(x,[t]))$ is a solution to problem (3.2)–(3.5), the function $v^*(x,t)$ defined as $v^*(x,t) = \lambda_s^*(x) + \tilde{v}_s^*(x,t)$, $(x,t) \in \Omega_s$, $s \in \mathbb{Z}$, is a solution to Problem 1_α and the estimate (3.12) holds true.

Let us now prove the uniqueness of the solution. Suppose that Problem 1_α has two solutions, $v^*(x,t)$ and $v^{**}(x,t)$. Then the corresponding systems of pairs $(\lambda^*(x), \tilde{v}^*(x,[t]))$ and $(\lambda^{**}(x), \tilde{v}^{**}(x,[t]))$ are the solutions to the boundary value problem with parameter (3.2)–(3.5) and

$$\|\tilde{v}^* - \tilde{v}^{**}\|_2 \leq \big(e^\theta - 1\big)\|\lambda^*(x) - \lambda^{**}(x)\|_1,$$

$$\|\lambda^* - \lambda^{**}\|_1 \leq q_\nu\big(\tilde{h}(\theta)\big)\|\lambda^* - \lambda^{**}\|_1, \qquad q_\nu\big(\tilde{h}(\theta)\big) < 1.$$

Hence $\lambda^*(x) = \lambda^{**}(x)$, $\tilde{v}^*(x,[t]) = \tilde{v}^{**}(x,[t])$, or $v^*(x,t) = v^{**}(x,t)$. Theorem 1 is proved. □

Note that under conditions of Theorem 1 it is not difficult to obtain the following estimate for the derivative of the solution $v^*(x,t)$ with respect to t:

$$\|v_t^*\|_\alpha \leq \|v^*\|_* + \|F\|_\alpha \leq \big(\tilde{M} + 1\big)\|F\|_\alpha, \tag{3.16}$$

$\tilde{M} = \gamma_\nu(\tilde{h}(\theta))[\frac{e^\theta M(\tilde{h}(\theta))}{1-q_\nu(\tilde{h}(\theta))}\frac{\theta^\nu}{\nu!} + \sum_{j=1}^{\nu}\frac{\theta^j}{j!}] + M(\tilde{h}(\theta))$.

This estimate shows that $v_t^*(x,t) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$ if $F(x,t) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$.

Letting $\nu \to \infty$ in (3.9) and taking into account

$$\|G_\nu\big(\tilde{v}^*, \tilde{h}(\theta), x\big)\|_1 \leq \frac{\theta^\nu}{\nu!}\|\tilde{v}^*(x,[t])\|_2,$$

we get that $\lambda^*(x) \in C([0,\omega], m_n)$ satisfies the equation

$$\frac{1}{\theta}Q_*(\tilde{h}(\theta), x)\lambda(x) = -F_*(A, F, \tilde{h}(\theta), x), \tag{3.17}$$

where $Q_*(\tilde{h}(\theta), x) = \lim\limits_{\nu\to\infty} Q_\nu(\tilde{h}(\theta), x)$, $F_*(A, F, \tilde{h}(\theta), x) = \frac{1}{\theta}\lim\limits_{\nu\to\infty} F_\nu(\tilde{h}(\theta), x)$.

The existence of a solution to Problem 1_α is equivalent to the existence of that to Eq. (3.17). In fact, the solution $\lambda(x) = (\ldots, \lambda_s(x), \lambda_{s+1}(x), \ldots)' \in C([0,\omega], m_n)$ to Eq. (3.17) coincides with the values of the solution to Problem 1_α at the partition points.

4. Singular boundary value problem for the system of hyperbolic equations

Let us now return to problem (1), (2). The following statement holds true.

Theorem 2. *Suppose that the following conditions hold:*

(i) $\|A(x,t)\| \leq \alpha(t)$, *where* $\alpha(t)$ *is a continuous and positive on R function satisfying (3) and (4);*
(ii) *the columns of* $B(x,t)$ *belong to* $C_*(\Omega^*, \mathbb{R}^n)$*;*
(iii) *the columns of* $C(x,t)$ *and* $f(x,t)$ *belong to* $C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$*;*
(iv) $\psi(t) \in C_*(\mathbb{R}, \mathbb{R}^n)$, $\dot{\psi}(t) \in C_{*,1/\alpha}(\mathbb{R}, \mathbb{R}^n)$*;*
(v) *the conditions of Theorem 1 are met.*

Then the singular boundary value problem for the system of hyperbolic Eq. (1), (2) has a unique classical solution $u^*(x,t) \in C_*(\Omega^*, \mathbb{R}^n)$.

Proof. It follows from conditions (i)-(iv) that

1) $B(x,t) = (b_{ij}(x,t))_{i,j=1}^n$, $\widetilde{b}_j(x,t) = (b_{1j}(x,t), b_{2j}(x,t), \dots, b_{nj}(x,t))'$, $j = 1,2,.,n$,

$$||\widetilde{b}_j||_* = \sup_{(x,t)\in\Omega^*} ||\widetilde{b}_j(x,t)|| = \sup_{(x,t)\in\Omega^*} \max_{i=\overline{1,n}} |b_{ij}(x,t)| \le \beta_0, \quad \beta_0 > 0;$$

2) $C(x,t) = (c_{ij}(x,t))_{i,j=1}^n$, $\widetilde{c}_j(x,t) = (c_{1j}(x,t), c_{2j}(x,t), \dots, c_{nj}(x,t))'$, $j = 1,2,.,n$,

$$||\widetilde{c}_j||_\alpha = \sup_{(x,t)\in\Omega^*} \|\widetilde{c}_j(x,t)/\alpha(t)\| \le \delta_0, \quad \delta_0 > 0.$$

Let us consider problem (2.1), (2.2) which is equivalent to problem (1), (2). Under condition (v), taking into account 1), 2) we can conclude that problem (2.1) has a unique solution $v(x,t) \in C_*(\Omega^*, \mathbb{R}^n)$ whenever $F(x,t,w(x,t),u(x,t)) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$. The following estimates are valid for $v(x,t)$ and its derivative:

$$\sup_{t\in\mathbb{R}} \|v(x,t)\| \le \|v\|_* \le \tilde{M}\|F\|_\alpha \le \tilde{M}\{\beta_0\|w\|_\alpha + \delta_0\|u\|_* + \|f\|_\alpha\}, \tag{4.1}$$

$$\sup_{t\in\mathbb{R}} \|v_t(x,t)/\alpha(t)\| \le \|v_t\|_\alpha \le \|v\|_* + \|F\|_\alpha \le \left[\tilde{M}+1\right]\|F\|_\alpha \le \left[\tilde{M}+1\right]\{\beta_0\|w\|_\alpha + \delta_0\|u\|_* + \|f\|_\alpha\}. \tag{4.2}$$

Hence, from integral relations (2.2) and inequalities (4.1), (4.2) we get

$$\sup_{t\in\mathbb{R}} ||u(x,t)|| \le \sup_{t\in\mathbb{R}} ||\psi(t)|| + \int_0^x \sup_{(\xi,t)\in\Omega_x^*} ||v(\xi,t)|| d\xi \le$$

$$\le \|\psi\|_* + \tilde{M}\int_0^x \left\{\beta_0 \sup_{(\xi,t)\in\Omega_x^*} \|\frac{w(\xi,t)}{\alpha(t)}\| + \delta_0 \sup_{(\xi,t)\in\Omega_x^*} \|u(\xi,t)\| + \sup_{(\xi,t)\in\Omega_x^*} \|\frac{f(\xi,t)}{\alpha(t)}\|\right\} d\xi, \tag{4.3}$$

$$\sup_{t\in\mathbb{R}} ||w(x,t)/\alpha(t)|| \le \sup_{t\in\mathbb{R}} \|\dot{\psi}(t)/\alpha(t)\| + \int_0^x \sup_{(\xi,t)\in\Omega_x^*} ||v_t(\xi,t)/\alpha(t)|| d\xi \le$$

$$\le \|\dot{\psi}\|_\alpha + \left[\tilde{M}+1\right]\int_0^x \left\{\beta_0 \sup_{(\xi,t)\in\Omega_x^*} \|\frac{w(\xi,t)}{\alpha(t)}\| + \delta_0 \sup_{(\xi,t)\in\Omega_x^*} \|u(\xi,t)\| + \sup_{(\xi,t)\in\Omega_x^*} \|\frac{f(\xi,t)}{\alpha(t)}\|\right\} d\xi. \tag{4.4}$$

Here $\Omega_x^* = [0,x] \times (-\infty, +\infty)$, $x \in [0,\omega]$.

The Volterra integral inequalities (4.3), (4.4) imply that there exist functions $u^*(x,t) \in C_*(\Omega^*, \mathbb{R}^n)$ and $w^*(x,t) \in C_{*,1/\alpha}(\Omega^*, \mathbb{R}^n)$ for which the following inequality holds:

$$\max\left(\|u^*\|_*, \|w^*\|_\alpha\right) \le \left(1 + \left[\tilde{M}+1\right]\omega\right) e^{\left[\tilde{M}+1\right](\beta_0+\delta_0)\omega} \max\left(\|f\|_\alpha, \|\psi\|_*, \|\dot{\psi}\|_\alpha\right). \tag{4.5}$$

From (4.1), (4.3) and (4.5) we have

$$\|v^*\|_* \le \tilde{M}\cdot K \cdot \max\left(\|f\|_\alpha, \|\psi\|_*, \|\dot{\psi}\|_\alpha\right), K = (\beta_0+\delta_0)\left(1 + \left[\tilde{M}+1\right]\omega\right) e^{\left[\tilde{M}+1\right](\beta_0+\delta_0)\omega} + 1 \tag{4.6}$$

$$\|v_t^*\|_\alpha \le \left[\tilde{M}+1\right]\cdot K \cdot \max\left(\|f\|_\alpha, \|\psi\|_*, \|\dot{\psi}\|_\alpha\right). \tag{4.7}$$

A solution to problem (1), (2) can be found by the following algorithm.

By solving problem (2.1) for $w(x,t) = \dot{\psi}(t)$ and $u(x,t) = \psi(t)$, we find $v^{(0)}(x,t)$. Then, from the integral relations (2.2), setting $v(x,t) = v^{(0)}(x,t)$ and $v_t(x,t) = v_t^{(0)}(x,t)$, we determine $u^{(0)}(x,t)$ and $w^{(0)}(x,t)$.

If $u^{(p-1)}(x,t)$ and $w^{(p-1)}(x,t)$ are known, we find $v^{(p)}(x,t)$, $p = 1,2,\dots,$ as solutions to problem (2.1) with $w(x,t) = w^{(p-1)}(x,t)$ and $u(x,t) = u^{(p-1)}(x,t)$. Then, from the integral relations (2.2), setting $v(x,t) = v^{(p)}(x,t)$ and $v_t(x,t) = v_t^{(p)}(x,t)$, we find $u^{(p)}(x,t)$ and $w^{(p)}(x,t)$.

In each step of the algorithm, for fixed $w(x,t)$ and $u(x,t)$, we get problem (3.1). The conditions of Theorem 1 guarantee the existence of a unique solution to Problem 1_α and the fulfilment of estimate (3.12). The unique solvability of problem 1_α in turn ensures convergence of the algorithm. The triple $\{v^*(x,t), u^*(x,t), w^*(x,t)\}$, being the limit of the sequence of function triples

$\{v^{(p)}(x,t), u^{(p)}(x,t), w^{(p)}(x,t)\}$ as $p \to \infty$, is a solution to Problem (2.1),(2.2). Its uniqueness can be proved by contradiction. Since problems (2.1), (2.2) and (1), (2) are equivalent, the classical solution $u^*(x,t)$ to problem (1), (2) is unique and belongs to the space $C_*(\Omega^*, \mathbb{R}^n)$. Theorem 2 is proved. □

Conclusion

In this paper, we have obtained conditions for the existence of a solution to a boundary value problem for a system of second order evolution equations of hyperbolic type with matrix $A(x,t)$ tending to zero as $t \to \mp\infty$. The coefficient $C(x,t)$, the right-hand side $f(x,t)$ of the system, and the derivative $\dot{\psi}(t)$ of the boundary function belong to a space of bounded functions with a weight that is chosen taking into account the behavior of $A(x,t)$ as $t \to \mp\infty$. The problem in question is reduced to an equivalent problem consisting of singular boundary value problems for a family of systems of first order ordinary differential equations and integral relations. We have established conditions for the existence of a bounded in a strip solution to a family of systems of ordinary differential equations with the matrix tending to zero as $t \to \mp\infty$ and the right-hand side bounded with a weight. The results and methods of this paper can be extended to nonlinear evolution equations of hyperbolic type and families of second order nonlinear evolution equations and can be used in the study of application problems.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Anar T. Assanova: Conceptualization, Methodology. **Roza E. Uteshova:** Investigation, Writing - review & editing.

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