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A MULTI-POINT PROBLEM FOR A SYSTEM OF DIFFERENTIAL EQUATIONS WITH PIECEWISE-CONSTANT ARGUMENT OF GENERALIZED TYPE AS A NEURAL NETWORK MODEL

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Key words: differential equations with piecewise-constant argument of generalized type, neural network model, multi-point boundary value problem, solvability criteria, algorithms of parameterization method.

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Abstract. We consider a system of ordinary differential equations with piecewise-constant argument of generalized type. An interval is divided into N parts, the values of a solution at the interior points of the subintervals are considered as additional parameters, and a system of ordinary differential equations with piecewise-constant argument of generalized type is reduced to the Cauchy problems on the subintervals for linear system of ordinary differential equations with parameters. Using the solutions to these problems, new general solutions to system of differential equations with piecewise-constant argument of generalized type are introduced and their properties are established. Based on the general solution, boundary condition, and continuity conditions of a solution at the interior points of the partition, the system of linear algebraic equations with respect to parameters is composed. Its coefficients and right-hand sides are found by solving the Cauchy problems for a linear system of ordinary differential equations on the subintervals. It is shown that the solvability of boundary value problems is equivalent to the solvability of composed systems. Methods for solving boundary value problems are proposed, which are based on the construction and solving of these systems.

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1 Introduction and statement of problem

It is well known that mathematical modeling of processes with discontinuity effects has necessitated the need to develop the theory of differential equations with discontinuities. An important class of such equations is comprised of differential equations with a piecewise constant argument (DEPCA). The study of DEPCA was initiated by Busenberg, Cooke, Shah, and Wiener [22], [19], [39]. The problems of the existence and uniqueness of solutions to DEPCA, their oscillations and stability, integral manifolds and periodic solutions have been extensively discussed by many authors [33], [20], [34], [40], [23], [38], [15], [16], [14].

When modeling DEPCA, the deviation of the argument, taken as the greatest integer function, is always constant and equal to one. But this approach can contradict real phenomena. The generalization of DEPCA has been undertaken by M.U.Akhmet [1], [2], [3], [4]. In his works the greatest integer function as deviating argument was replaced by an arbitrary piecewise constant function. Thus, differential equations with piecewise constant argument of generalized type (DEPCAG) are more suitable for modeling and solving various applied problems, including areas of neural networks, discontinuous dynamical systems, hybrid systems, etc. To date, the theory of DEPCAG on the entire axis has been developed and their applications have been implemented. The results have been extended to periodic impulse systems of DEPCAG [5], [6], [11], [9], [10], [21], [7]. Note that an electronic neural networks were modeled as differential equations with piecewise constant arguments of generalized type [18], [11], [9]. By reducing these equations to an equivalent integral equation, some new stability conditions are obtained.

Along with the study of various properties of DEPCA, a number of authors investigated the problems of solvability and construction of solutions to boundary value problems for these equations on a finite interval [35], [37], [24], [8].

For DEPCAG, however, the problems of solvability of boundary value problems on a finite interval still remain open.

This issue can be resolved by developing constructive methods.

So, on $[0,T]$, we consider the following multi-point boundary value problem for a system of DEPCAG:

$$\frac{dx}{dt} = A(t)x + A_0(t)x(\gamma(t)) + f(t), \qquad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{1.1}$$

$$\sum_{i=0}^{N} B_i x(\theta_i) = d, \qquad d \in \mathbb{R}^n. \tag{1.2}$$

Here $x(t) = col(x_1(t), x_2(t), ..., x_n(t))$ is the unknown function, $(n \times n)$ matrices $A(t)$, $A_0(t)$ and n-vector $f(t)$ are continuous on $[0,T]$;
$\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$; $\theta_j \le \zeta_j \le \theta_{j+1}$ for all $j = 0, 1, \ldots, N-1$; $0 = \theta_0 < \theta_1 < \ldots < \theta_{N-1} < \theta_N = T$; B_i are constant $(n \times n)$ matrices, $i = \overline{0, N}$, and d is a constant vector.

The aim of the present paper is to develop a constructive method for investigating and solving the boundary value problem, including an algorithm for finding a solution to problem (1.1), (1.2) as well.

To this end, we use a new concept of general solution and Dzhumabaev's parametrization method [25], [26]. This concept of general solution has been introduced for the linear Fredholm integro-differential equation in [27] and for the linear loaded differential equation and a family of such equations in [28], [29]. New general solutions are also introduced to ordinary differential equations and their properties are established in [30]. Results are developed to nonlinear Fredholm integro-differential equations [31], [32] and to problems with a parameter for integro-differential equations [13]. Based on the general solution methods for solving boundary value problems are proposed.

The paper is organized as follows.

The interval $[0,T]$ is divided into N parts according to the partition $\Delta_N : \theta_0 = 0 < \theta_1 < \theta_2 < ... < \theta_N = T$, and the Δ_N general solution to a linear system of differential equation with a piecewise-constant argument of generalized type is introduced. The Δ_N general solution, denoted by $x(\Delta_N, t, \lambda)$, contains an arbitrary vectors $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbb{R}^{nN}$. Using $x(\Delta_N, t, \lambda)$, we establish solvability criteria of considered problem and propose an algorithm for finding its solution.

A function $x^*(t) : [0,T] \to \mathbb{R}^n$ is a solution to problem (1.1), (1.2) if:

(i) $x^*(t)$ is continuous on $[0,T]$;

(ii) $x^*(t)$ is differentiable on $[0,T]$ with the possible exception of the points θ_j, $j = \overline{0, N-1}$, at which the one-sided derivatives exist;

(iii) $x^*(t)$ satisfies the system of equations (1.1) on each interval (θ_j, θ_{j+1}), $j = \overline{0, N-1}$; at the points θ_j, $j = \overline{0, N-1}$, system (1) is satisfied by the right-hand derivative of $x^*(t)$;

(iv) $x^*(t)$ satisfies boundary condition (1.2) at $t = \theta_i$, $i = \overline{0, N}$.

2 Scheme of the method and Δ_N general solution

Let Δ_N denote the partition of the interval $[0, T)$ by points $t = \theta_r$, $r = \overline{1, N-1}$: $[0,T) = \bigcup\limits_{r=1}^{N} [\theta_{r-1}, \theta_r)$.

We define the following spaces:

$C([0,T], \mathbb{R}^n)$ is the space of all continuous functions $x : [0,T] \to \mathbb{R}^n$ with the norm

$$\|x\|_1 = \max_{t\in[0,T]} ||x(t)|| = \max_{t\in[0,T]} \max_{i=\overline{1,n}} |x_i(t)|;$$

$C([0,T], \Delta_N, \mathbb{R}^{nN})$ is the space of function systems $x[t] = (x_1(t), x_2(t), \dots, x_N(t))$, where x_r : $[\theta_{r-1}, \theta_r) \to \mathbb{R}^n$ are continuous functions that have finite left-hand limits $\lim\limits_{t\to\theta_r-0} x_r(t)$ for all $r = \overline{1, N}$, with the norm

$$\|x[\cdot]\|_2 = \max_{r=\overline{1,N}} \sup_{t\in[\theta_{r-1},\theta_r)} |x_r(t)|.$$

Denote by $x_r(t)$ the restriction of a function $x(t)$ to the rth interval $[\theta_{r-1}, \theta_r)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$.

Then the function system $x[t] = (x_1(t), x_2(t), \dots, x_N(t))$ belongs to $C([0,T], \Delta_N, \mathbb{R}^{nN})$, and its elements $x_r(t)$, $r = \overline{1, N}$, satisfy the following system of ordinary differential equations with piecewise-constant argument of generalized type

$$\frac{dx_r}{dt} = A(t)x_r(t) + A_0(t)x_r(\zeta_{r-1}) + f(t), \qquad t \in [\theta_{r-1}, \theta_r), \quad r = \overline{1, N}. \tag{2.1}$$

In (2.1) we take into account that $\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$.

We introduce additional parameters $\lambda_r = x_r(\zeta_{r-1})$ for all $r = \overline{1, N}$. Making the substitution $z_r(t) = x_r(t) - \lambda_r$ on every r-th interval $[\theta_{r-1}, \theta_r)$, we obtain the system of ordinary differential equations with parameters

$$\frac{dz_r}{dt} = A(t)(z_r(t) + \lambda_r) + A_0(t)\lambda_r + f(t), \qquad t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1, N}, \tag{2.2}$$

and initial conditions

$$z_r(\zeta_{r-1}) = 0, \qquad r = \overline{1, N}. \tag{2.3}$$

Problems (2.2), (2.3) are Cauchy problems for system of ordinary differential equations with parameters on the intervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$. For any fixed $\lambda_r \in \mathbb{R}^n$ and r, the Cauchy problem (2.2), (2.3) has a unique solution $z_r(t, \lambda_r)$, and the function system $z[t, \lambda] = (z_1(t, \lambda_1), z_2(t, \lambda_2), \dots, z_N(t, \lambda_N))$ belongs to $C([0,T], \Delta_N, \mathbb{R}^{nN})$.

The function system $z[t, \lambda]$ is referred to as a solution to Cauchy problems with parameters (2.2), (2.3). If a function system $\widetilde{x}[t] = (\widetilde{x}_1(t), \widetilde{x}_2(t), ..., \widetilde{x}_N(t))$ belongs to $C([0,T], \Delta_N, \mathbb{R}^{nN})$, and the functions $\widetilde{x}_r(t)$, $r = \overline{1, N}$, satisfy equations (2.1), then the function system $z[t, \widetilde{\lambda}] = (z_1(t, \widetilde{\lambda}_1), z_2(t, \widetilde{\lambda}_2)$, ..., $z_N(t, \widetilde{\lambda}_N))$ with the elements $z_r(t, \widetilde{\lambda}_r) = \widetilde{x}_r(t) - \widetilde{\lambda}_r$, $\widetilde{\lambda}_r = \widetilde{x}_r(\zeta_{r-1})$, $r = \overline{1, N}$, is a solution to the Cauchy problems with parameters (2.2), (2.3) for $\lambda_r = \widetilde{\lambda}_r$, $r = \overline{1, N}$. Conversely, if a function system $z[t, \lambda^*] = (z_1(t, \lambda_1^*), z_2(t, \lambda_2^*), \dots, z_N(t, \lambda_N^*))$ is a solution to problems (2.2), (2.3) for $\lambda_r = \lambda_r^*$, $r = \overline{1, N}$, then the function system $x^*[t] = (x_1^*(t), x_2^*(t), \dots, x_N^*(t))$ with $x_r^*(t) = \lambda_r^* + z_r(t, \lambda_r^*)$, $r = \overline{1, N}$, belongs to $C([0,T], \Delta_N, \mathbb{R}^{nN})$, and the functions $x_r^*(t)$, $r = \overline{1, N}$, satisfy system of equations (2.1).

Let us now introduce a new general solution to the system of ordinary differential equations with piecewise-constant argument of generalized type (2.1).

Definition 1. Let $z[t,\lambda] = (z_1(t,\lambda_1), z_2(t,\lambda_2), \ldots, z_N(t,\lambda_N))$ be the solution to the Cauchy problems (2.2), (2.3) for the parameters $\lambda = (\lambda_1, \lambda_2, ..., \lambda_N) \in \mathbb{R}^{nN}$. Then the function $x(\Delta_N, t, \lambda)$, given by the equalities

$x(\Delta_N, t, \lambda) = \lambda_r + z_r(t, \lambda_r)$, for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$, and

$x(\Delta_N, T, \lambda) = \lambda_N + \lim\limits_{t \to T-0} z_N(t, \lambda_N)$,

is called the Δ_N general solution to system of equations (2.1).

As follows from Definition 2.1, the Δ_N general solution depends on N arbitrary vectors $\lambda_r \in \mathbb{R}^n$ and satisfies system of equations (2.1) for all $t \in (0, T)\backslash\{\theta_p, p = \overline{1, N-1}\}$.

Take $X_r(t)$, a fundamental matrix of the ordinary differential equation

$$\frac{dz_r}{dt} = A(t)z_r(t), \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N},$$

and write down the solutions to the Cauchy problems with parameters (2.2), (2.3) in the form:

$$z_r(t) = X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)[A(\tau) + A_0(\tau)]d\tau\lambda_r + X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)f(\tau)d\tau,$$

$$t \in [\theta_{r-1}, \theta_r), \qquad r = \overline{1, N}.$$

Consider the Cauchy problems on the subintervals

$$\frac{dx}{dt} = A(t)x + P(t), \qquad x(\zeta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N}, \tag{2.4}$$

where $P(t)$ is a square matrix or a vector of dimension 2, continuous on $[0, T]$, $\theta_{r-1} \le \zeta_{r-1} \le \theta_r$ for all $r = 1, 2, ..., N$. Denote by $A_r(P, t)$ a unique solution to Cauchy problem (2.4) on each rth interval. The uniqueness of the solution to the Cauchy problem for linear ordinary differential equations yields

$$A_r(P, t) = X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)P(\tau)d\tau, \qquad t \in [\theta_{r-1}, \theta_r], \qquad r = \overline{1, N}.$$

Therefore, we can represent the Δ_N general solution to system of equations (2.1) in the form:

$$x(\Delta_N, t, \lambda) = \lambda_p + A_p(A + A_0, t)\lambda_p + A_p(f, t), \quad t \in [\theta_{p-1}, \theta_p), \quad p = \overline{1, N-1}, \tag{2.5}$$

$$x(\Delta_N, t, \lambda) = \lambda_N + A_N(A + A_0, t)\lambda_N + A_N(f, t), \qquad t \in [\theta_{N-1}, \theta_N]. \tag{2.6}$$

The following statement affirms the function $x(\Delta_N, t, \lambda)$ as a "general solution".

Theorem 2.1. *Let a piecewise continuous on* $[0, T]$ *function* $\widetilde{x}(t)$ *with the possible discontinuity points* $t = \theta_p$, $p = \overline{1, N-1}$, *be given, and* $x(\Delta_N, t, \lambda)$ *be the* Δ_N *general solution to system of equations* (2.1). *Suppose that the function* $\widetilde{x}(t)$ *has a continuous derivative and satisfies system of equations* (2.1) *for all* $t \in (0, T)\backslash\{\theta_p, p = \overline{1, N-1}\}$. *Then there exists a unique* $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, ..., \widetilde{\lambda}_N) \in \mathbb{R}^{nN}$ *such that the equality* $x(\Delta_N, t, \widetilde{\lambda}) = \widetilde{x}(t)$ *holds for all* $t \in [0, T]$.

The proof of this theorem is quite simple. Therefore, we do not present it.

Corollary 2.1. *Let* $x^*(t)$ *be a solution to system of equations* (2.1) *and* $x(\Delta_N, t, \lambda)$ *be the* Δ_N *general solution to system of equations* (2.1). *Then there exists a unique* $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$ *such that the equality* $x(\Delta_N, t, \lambda^*) = x^*(t)$ *holds for all* $t \in [0, T]$.

If $x(t)$ is a solution to system of equations (2.1), and $x[t] = (x_1(t), x_2(t), ..., x_N(t))$ is a function system composed of its restrictions to the subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$, then the equations

$$\lim_{t \to \theta_p - 0} x_p(t) = x_{p+1}(\theta_p), \qquad p = \overline{1, N-1}, \tag{2.7}$$

hold. These equations are the continuity conditions for the solution to system of equations (2.1) at the interior points of the partition Δ_N.

Theorem 2.2. *Let a function system $x[t] = (x_1(t), x_2(t), ..., x_N(t))$ belong to $C([0,T], \Delta_N, \mathbb{R}^{nN})$. Assume that the functions $x_r(t)$, $r = \overline{1, N}$, satisfy system of equations* (2.1) *and continuity conditions* (2.7)*. Then the function $x^*(t)$, given by the equalities*

$x^*(t) = x_r(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $\quad r = \overline{1, N}$,

and $\quad x^*(T) = \lim\limits_{t \to T-0} x_N(t)$,

is continuous on $[0,T]$, continuously differentiable on $(0,T)$ and satisfies system of equations (2.1).

Proof. Equations (2.7), the equality $x^*(T) = \lim\limits_{t \to T-0} x_N(t)$, and belonging of $x[t] = (x_1(t), x_2(t), ..., x_N(t))$ to $C([0,T], \Delta_N, \mathbb{R}^{nN})$ ensure continuity of the function $x^*(t)$ on the interval $[0,T]$. Since the functions $x_r(t)$, $r = \overline{1, N}$, satisfy system of equations (3), the function $x^*(t)$ has continuous derivative and satisfies system of equations (2.1) for all $t \in [0,T] \backslash \{\theta_p, p = \overline{1, N-1}\}$. The existence and continuity of the derivative of the function $x^*(t)$ at the points $t = \theta_p$, $p = \overline{1, N-1}$, follow from the relations:

$$\lim_{t \to \theta_p - 0} \dot{x}^*(t) = A(\theta_p) x^*(\theta_p) + A_0(\theta_p) x^*(\zeta_{p-1}) + f(\theta_p) = \lim_{t \to \theta_p + 0} \dot{x}^*(t), \ \ p = \overline{1, N-1}.$$

Hence the function $x^*(t)$ satisfies system of equations (2.1) at the interior points of the partition Δ_N as well.

□

3 Main results and algorithm

The Δ_N general solution allows us to transfer the solvability of a multi-point boundary value problem to the solvability of a system of linear algebraic equations with respect to arbitrary vectors $\lambda_r \in \mathbb{R}^2$, $r = \overline{1, N}$.

Substituting the suitable expressions of Δ_N general solution (2.5), (2.6) into the multi-point condition (1.2) and continuity conditions (2.7), we obtain the system of linear algebraic equations

$$\sum_{i=0}^{N} B_i \Big\{ I + A_i(A + A_0, \theta_i) \Big\} \lambda_i = d - \sum_{i=0}^{N} B_i A_i(f, \theta_i), \tag{3.1}$$

$$\begin{aligned} &\Big\{ I + A_p(A, \theta_p) \Big\} \lambda_p - \Big\{ I + A_{p+1}(A + A_0, \theta_p) \Big\} \lambda_{p+1} \\ &\quad = -A_p(f, \theta_p) + A_{p+1}(f, \theta_p), \qquad p = \overline{1, N-1}, \end{aligned} \tag{3.2}$$

where I is the unit matrix of dimension n.

Denote by $Q_*(\Delta_N)$ $nN \times nN$ matrix corresponding to the left-hand side of system (3.1), (3.2) and write the system as

$$Q_*(\Delta_N) \lambda = -F_*(\Delta_N), \qquad \lambda \in \mathbb{R}^{nN}, \tag{3.3}$$

where $F_*(\Delta_N) = \Big(-d + \sum\limits_{i=0}^{N} B_i A_i(f, \theta_i), A_1(f, \theta_1) - A_2(f, \theta_1),$

$A_2(f, \theta_2) + A_3(f, \theta_2), ..., A_{N-1}(f, \theta_{N-1}) + A_N(f, \theta_{N-1}) \Big) \in \mathbb{R}^{nN}.$

For any partition Δ_N, Theorems 2.1 and 2.2 ensure the validity of the next assertion.

Lemma 3.1. *If $x^*(t)$ is a solution to multi-point problem* (1.1), (1.2) *and $\lambda_r^* = x^*(\zeta_{r-1})$, $r = \overline{1,N}$, then the vector $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$ is a solution to system* (3.3). *Conversely, if $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_N) \in \mathbb{R}^{nN}$ is a solution to system* (3.3) *and $z[t, \tilde{\lambda}] = (z_1(t, \tilde{\lambda}_1), z_2(t, \tilde{\lambda}_2), \ldots, z_N(t, \tilde{\lambda}_N))$ is the solution to Cauchy problems* (2.2), (2.3) *for the parameter $\tilde{\lambda} \in \mathbb{R}^{nN}$, then the function $\tilde{x}(t)$ given by the equalities $\tilde{x}(t) = \tilde{\lambda}_r + z_r(t, \tilde{\lambda}_r)$, $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$, and $\tilde{x}(T) = \tilde{\lambda}_N + \lim\limits_{t \to T-0} z_N(t, \tilde{\lambda}_N)$, is a solution to multi-point problem* (1.1), (1.2).

Definition 2. The multi-point boundary value problem (1.1), (1.2) is called uniquely solvable if for any pair $(f(t), d)$, with $f(t) \in C([0,T], \mathbb{R}^n)$ and $d \in \mathbb{R}^n$, it has a unique solution.

Lemma 3.1 and well-known theorems of linear algebra imply the following two statements.

Theorem 3.1. *The multi-point boundary value problem* (1.1), (1.2) *is solvable if and only if the vector $F_*(\Delta_N)$ is orthogonal to the kernel of the transposed matrix $(Q_*(\Delta_N))'$, i.e. if and only if the equality*

$$(F_*(\Delta_N), \eta) = 0$$

is valid for all $\eta \in Ker(Q_*(\Delta_N))'$, where $(\cdot,\cdot)$ is the inner product in $\mathbb{R}^{2N}$.

Theorem 3.2. *The multi-point boundary value problem* (1.1), (1.2) *is uniquely solvable if and only if $nN \times nN$ matrix $Q_*(\Delta_N)$ is invertible.*

So, by Theorems 3.1 and 3.2 it follows that the solvability of multi-point boundary value problem (1.1), (1.2) is equivalent to the solvability of system of algebraic equations (3.3). This system composed by solutions of Cauchy problems (2.2), (2.3), of multi-point condition (1.2) and continuity condition (2.7).

Based on the results of Section 3, we offer the following algorithm for finding a solution to the linear multi-point boundary value problem (1.1), (1.2).

Algorithm.

Step 1. Solve the Cauchy problems on the subintervals

$$\frac{dz}{dt} = A(t)z + A(t) + A_0(t), \qquad z(\zeta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r],$$

$$\frac{dz}{dt} = A(t)z + f(t), \qquad z(\zeta_{r-1}) = 0, \qquad t \in [\theta_{r-1}, \theta_r],$$

and find $A_r(A + A_0, \theta_r)$ and $A_r(f, \theta_r)$, $r = \overline{1,N}$. Here $\theta_{r-1} \leq \zeta_{r-1} \leq \theta_r$ for all $r = 1, 2, ..., N$.

Step 2. Using the found matrices and vectors compose the system of linear algebraic equations (12).

Step 3. Solve the constructed system and find $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_N^*) \in \mathbb{R}^{nN}$. Note that the elements of λ^* are the values of the solution to multi-point problem (1.1), (1.2) at the interior points of the subintervals: $\lambda_r^* = u^*(\zeta_{r-1})$, $r = \overline{1,N}$.

Step 4. Solve the Cauchy problems

$$\frac{dz}{dt} = A(t)z + f(t), \qquad z(\zeta_{r-1}) = \lambda_r^*, \qquad t \in [\theta_{r-1}, \theta_r),$$

and define the values of the solution $x^*(t)$ at the remaining points of the subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$.

The function $x^*(t)$ is a solution to original multi-point problem (1.1), (1.2).

As it follows from Lemma 3.1, any solution to system (3.3) determines the values of the solution to problem (1.1), (1.2) at the left end-points of the subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1,N}$.

The accuracy of the algorithm proposed depends on the accuracy of computing the coefficients and right-hand sides of system of algebraic equations (3.3).

The Cauchy problem for a system of ordinary differential equations is the principal auxiliary problem in the offered algorithm. By choosing an approximate method for solving that problem, we obtain an approximate method for solving the multi-point boundary value problem (1.1), (1.2). The solution of the Cauchy problems by numerical methods leads to numerical algorithms for solving multi-point problem (1.1), (1.2).

Remark 1. In the general case, the points $t = \theta_i$ in the multi-point condition may not coincide with the left-end points of the subintervals $[\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$. In this case, we can re-number all the points so that the points of the multi-point condition become the left-end points of the subintervals.

Conclusion. In the paper, we propose a new approach aimed at studying multi-point boundary value problems for systems of differential equations with a piecewise constant argument of generalized type. This method is based on a new concept of general solution of differential equations with piecewise constant argument of generalized type and Dzhumabaev's parametrization method. New general solution enables us to establish the qualitative properties of multi-point boundary value problems for systems of differential equations with a piecewise constant argument of generalized type and to develop algorithms for solving them. The algorithms are based on constructing and solving systems of linear algebraic equations in arbitrary vectors of new general solution. The results obtained can be used in a wide range of applications: problems for impulsive differential equations with a piecewise constant argument of generalized type; the theory of dynamical systems and neural networks; nonlocal problems for hyperbolic equations with a piecewise constant argument of generalized type, etc. [17], [36].

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