Research Article

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Fractional differential equations on Lie groups and submonoids

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Abstract: In the present work we define generic fractional differential operators on Lie groups and their submonoids. Criteria of solvability and Fredholmness for fractional differential equations are established and fundamental solutions in case of operators with constant coefficients are provided. In the second part, we solve six distinct Cauchy initial value problems for degenerate diffusion equations with fractional Hadamard type derivatives in time.

Keywords: time-fractional diffusion equation, the Kilbas–Saigo function, Cauchy–Dirichlet problem, Cauchy–Neumann problem, Cauchy problem.

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1 Introduction

Fractional calculus finds extensive applications in fields such as mathematical modeling, mechanics, physics, etc. (see [1, 3, 15, 23]). A considerable number of researchers have studied equations with different forms of fractional derivatives [19]. In recent years, there has been growing interest in studying linear and non-linear diffusion equations especially with Hadamard fractional derivatives, which were first defined in [14] (see, e.g., [19, 22, 25]). The solvability of degenerate diffusion equations with the time-fractional Caputo and Riemann–Liouville derivatives were investigated in [26, 28]. In [27], degenerate diffusion equations with the Hadamard fractional derivative are considered, the existence and uniqueness of solutions is proved, and behavior of weak solutions under specific initial and boundary conditions is investigated.

Fractional differential operators represent pseudodifferential (convolution) operators of fractional order. In Section 2 of the present paper, we define fractional differential operators (including Hadamard's fractional derivatives) on Lie groups. We study properties of these generic fractional derivatives and indicate criteria for the unique solvability of equations with fractional derivatives and constant coefficients. Moreover, we investigate equations with fractional derivatives and constant coefficients restricted to submonoids, where the theory of equations becomes much richer (e.g., Wiener–Hopf equations on the half-axes versus convolution equations on the axes). Criteria for Fredholm property, solvability and index theory are exposed (for more details see [6–9]).

In Section 3, we define and study the modified Hadamard type fractional derivatives $\mathfrak{D}_{\pm}$ on the Lie group $\{\mathbb{R}^+, \times\}$ and the Hadamard type fractional derivatives $\mathfrak{D}_{I,\pm}$ on the Lie group $\{\mathcal{I} = (-1, 1), \circ\}$. It is worth mentioning that in Lemma 3.2 the fundamental solutions to the Hadamard fractional differential

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operators are exposed

$$k_-^{-\alpha}(t) = \theta_I(t)\frac{e^{-i\pi\alpha}(-\ln t)^{\alpha-1}}{\Gamma(\alpha)},$$
$$k_+^{-\alpha}(t) = \theta_{(1,\infty)}(t)\frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}, \quad t \in \mathbb{R}^+.$$

In Section 4, we solve six different Cauchy initial boundary value problems for degenerate and non-degenerate diffusion equations of the following types:

$\mathfrak{D}^\alpha_{+,t}u(t,x) - \lambda(t-a)^\gamma\Delta_x u(t,x) = 0$ on the domain $\mathbb{R}^+_a \times \mathbb{R}^N$, $\mathbb{R}^+_a := (a,\infty)$;
$\mathfrak{D}^\alpha_{+,t}u(t,x) - \left(\log \frac{t}{a}\right)^\gamma \Delta_x u(t,x) = 0$ on the domain $\mathbb{R}^+_a \times \Omega$;
$\mathfrak{D}^\alpha_{+,t}u(t,x) - \lambda(t-a)^\gamma\Delta_x u(t,x) = 0$ on the set $(a,b] \times \mathbb{R}^N$;
$\mathfrak{D}^\alpha_{+,t}u(t,x) - \lambda(t-a)^\gamma\Delta_x u(t,x) = 0$ on the domain $(a,b] \times \Omega$;
$\mathfrak{D}^\alpha_{+,t}u(t,x) - \lambda\Delta_x u(t,x) = f(t,x)$ on the set $(a,b] \times \mathbb{R}^N$;
$\mathfrak{D}^\alpha_{+,t}u(t,x) - \lambda\Delta_x u(t,x) = f(t,x)$ on the set $(a,b] \times \Omega$.

Here, Ω is an open, bounded set in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. To prove the existence of solutions, we use the Fourier method, spectral properties of the Laplace operator Δ_x on the domain Ω and basic properties of the Kilbas–Saigo function $E_{\alpha,m,l}(z)$ (see Definition 3.4 and [19, Remark 5.1]) as well as Mittag–Leffler function $E_{\alpha,\beta}(z)$ (see Definition 3.4 and [19, pages 42,43]).

For the solution of diffusion equations with initial data on the Euclidean space $\mathbb{R}^N$, $N \geqslant 1$, we apply the Fourier transformation and the properties of convolution operators on $\mathbb{R}^N$.

2 Auxiliary results

In the present section, we expose some auxiliary results from [2, 6, 7].

Consider a Lie group $\{\boldsymbol{G},\circ\}$ with the group operation $x \circ y$. Then on $\{\boldsymbol{G},\circ\}$ we have a uniquely defined Haar measure $d\mu_{\boldsymbol{G}}$, the Fourier transformation $\mathcal{F}_{\boldsymbol{G}}$, the inverse Fourier transformation $\mathcal{F}^{-1}_{\boldsymbol{G}}$ and the generic differential operators (GDOs in short) $\mathfrak{D}_1,\dots,\mathfrak{D}_n$, generated by the vector fields from the corresponding Lie algebra. We assume that $\boldsymbol{G}$ is homomorphic to the Lie group $\{\mathbb{R}^n, x \circ y = x + y\}$, the dual group is then $\widehat{\boldsymbol{G}} = \mathbb{R}^n$ and, using pull-back operators associated with the group homomorphism $x(t) : \boldsymbol{G} \to \mathbb{R}^n$ and its inverse $t(x) : \mathbb{R}^n \to \boldsymbol{G}$, we define the Schwartz spaces of fast decaying smooth functions $\mathbb{S}(\boldsymbol{G})$ and its dual space of Schwartz distributions $\mathbb{S}'(\boldsymbol{G})$.

More details about Lie groups can be found, e.g., in the book [10].

It is well known that the Schwartz space $\mathbb{S}(\boldsymbol{G})$ endowed with standard seminorms and its dual space of Schwartz distributions $\mathbb{S}'(\boldsymbol{G})$ are invariant under the Fourier transformation $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ (the operators $\mathcal{F}^{\pm 1}$ are bounded in these spaces). Therefore, for a (complex-valued) symbol $a(\xi)$ from the Schwartz space of distributions $a \in \mathbb{S}'(\widehat{\boldsymbol{G}}) = \mathbb{S}'(\mathbb{R}^n)$, the associated convolution operator

$$W^0_{a,\boldsymbol{G}} := \mathcal{F}^{-1}_{\boldsymbol{G}} a \mathcal{F}_{\boldsymbol{G}} \;:\; \mathbb{S}(\boldsymbol{G}) \longrightarrow \mathbb{S}'(\boldsymbol{G})$$

is defined correctly. The notation $\mathfrak{M}_p(\widehat{\boldsymbol{G}})$ is used for the set of functions such that the convolution operator $W^0_{a,\boldsymbol{G}} : \mathbb{L}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}}) \to \mathbb{L}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$, $1 < p < \infty$, is bounded. Here, $\mathbb{L}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$, $1 < p < \infty$, is the Lebesgue space of complex-valued functions with the measure $\boldsymbol{d}\mu_{\boldsymbol{G}}$ and $\mathcal{L}(\mathfrak{B})$ denotes the algebra of all linear bounded operators on the Banach space $\mathfrak{B}$. $\mathfrak{M}_p(\widehat{\boldsymbol{G}})$ coincides, obviously, with the classical algebra of $\mathbb{L}_p$-multipliers on the Euclidean space $\mathfrak{M}_p(\mathbb{R}^n)$. Due to Plancherel's theorem on the isometrical isomorphism $(2\pi)^{\mp n/2}\mathcal{F}^{\pm 1}_{\boldsymbol{G}} : \mathbb{L}_2(\boldsymbol{G}) \to \mathbb{L}_2(\boldsymbol{G})$, the equality $\mathfrak{M}_2(\widehat{\boldsymbol{G}}) = \mathbb{L}_\infty(\mathbb{R}^n)$ holds: The convolution operator $W^0_{a,\boldsymbol{G}}$ is bounded in $\mathbb{L}_2(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$ if and only if $a \in \mathbb{L}_\infty(\mathbb{R}^n)$, and $\mathfrak{M}_p(\mathbb{R}^n) \subset \mathfrak{M}_2(\mathbb{R}^n) = \mathbb{L}_\infty(\mathbb{R}^n)$ for all $1 < p < \infty$ (see [2]).

Note that $\mathfrak{M}_p(\widehat{\boldsymbol{G}})$ is a Banach algebra, endowed with the norm

$$\|a\big|\mathfrak{M}_p(\widehat{\boldsymbol{G}})\| := \|W^0_{\boldsymbol{G},a}\big|\mathbb{L}_p(\widehat{\boldsymbol{G}})\|$$

due to the following property:

$$W^0_{\boldsymbol{G},a}W^0_{\boldsymbol{G},b} = W^0_{\boldsymbol{G},ab}, \quad a, b \in \mathfrak{M}_p(\widehat{\boldsymbol{G}}). \tag{2.1}$$

By $\boldsymbol{PC}_p(\widehat{\boldsymbol{G}}) = \boldsymbol{PC}_p(\mathbb{R}^n)$ we denote the closure of the algebra of piecewise-constant functions, which are constant on a finite set of non-intersecting open polytops (intersection of a finite number of half-spaces) in the norm of the algebra of multipliers $\mathfrak{M}_p(\mathbb{R}^n)$. Due to the well-known Hörmander's inequality

$$\|W^0_{\boldsymbol{G},a}\big|\mathbb{L}_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})\| \geqslant \|W^0_{\boldsymbol{G},a}\big|\mathbb{L}_2(\boldsymbol{G}, d\mu_{\boldsymbol{G}})\| = \sup_{\xi\in\widehat{\boldsymbol{G}}} |a(\xi)|$$

(cf. [16]), functions from $\boldsymbol{PC}_p(\mathbb{R}^n)$ have at most countable number of different radial limits at all points $x \in \mathbb{R}^n \bigcup\{\infty\}$.

Let $r \in \mathbb{R}$ and $\mathfrak{M}^r_p(\widehat{\boldsymbol{G}}) := \{\langle\xi\rangle^r a(\xi) : a \in \mathfrak{M}_p(\widehat{\boldsymbol{G}})\|$. $\boldsymbol{PC}^r_p(\widehat{\boldsymbol{G}})$ is defined similarly (symbols growing with the order r at the infinity).

For the class of matrix symbols and vector spaces, we use the same notations $\mathfrak{M}^r_p(\widehat{\boldsymbol{G}})$, $\boldsymbol{PC}^r_p(\widehat{\boldsymbol{G}})$, $\mathbb{L}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$, $\mathbb{GH}^s_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$ (see later).

Theorem 2.1 (see [7]). *Let $1 < p < \infty$. For an integro-differential convolution operator $W^0_{\boldsymbol{G},a}$ with the symbol $a \in \boldsymbol{PC}^r_p(\widehat{\boldsymbol{G}})$, there exists a kernel $\mathcal{K}_a$ (a distribution in general) such that the operator is written as an integral convolution with this kernel:*

$$W^0_{\boldsymbol{G},a}\varphi(x) := \mathcal{K}_a \star_{\boldsymbol{G}} \varphi(x) = \int\limits_{\boldsymbol{G}} \mathcal{K}_a(x\circ y^{-1})\varphi(y)d\mu_{\boldsymbol{G}}, \quad \varphi \in \mathbb{S}(\boldsymbol{G}). \tag{2.2}$$

Note that the convolution $\mathcal{K}_a \star_{\boldsymbol{G}} \varphi$ of a distribution $\mathcal{K}_a \in \mathbb{S}'(\boldsymbol{G})$ with a function $\varphi \in \mathbb{S}(\boldsymbol{G})$ is a correctly defined operation.

L. Hörmander proved the above theorem for the case $\{\boldsymbol{G}, \circ\} = \{\mathbb{R}^n, +\}$ (cf. [16]). In our case, the theorem remains valid due to the homeomorphism of $\boldsymbol{G}$ and $\mathbb{R}^n$,

Since the Lie groups $\boldsymbol{G}$ we consider are homomorphic to $\mathbb{R}^n$, the generic differential operators are

$$\mathfrak{D}_k := \boldsymbol{W}^0_{\boldsymbol{G},-i\xi_k} = \mathcal{F}^{-1}_{\boldsymbol{G}}(-i\xi_k)\mathcal{F}_{\boldsymbol{G}}, \quad k = 1, 2, \dots, n.$$

The following natural question arises: which spaces are best suited to consider solvability of the convolution integro-differential equation

$$\begin{gathered}\boldsymbol{W}^0_{\boldsymbol{G},a}\varphi(x) := \sum_{\leqslant n} c_\alpha\mathcal{D}^\alpha\varphi(x) + \sum_{|\beta|+|\gamma|\leqslant n} \mathcal{D}^\beta \int\limits_{\boldsymbol{G}} k_{\beta\gamma}(x\circ y^{-1})\mathcal{D}^\gamma\varphi(y)\boldsymbol{d}\mu_{\boldsymbol{G}}(y) = f(y), \\ k_{\beta\gamma} \in \mathbb{L}_1(\boldsymbol{G}, d\mu_{\boldsymbol{G}}), \quad \alpha, \beta, \gamma \in \mathbb{N}^m_0, \quad \mathbb{N}_0 := \{0, 1, \dots\}, \quad \mathfrak{D}^\alpha := \mathfrak{D}^{\alpha_1}_1 \cdots \mathfrak{D}^{\alpha_n}_n,\end{gathered} \tag{2.3}$$

with symbols of polynomial growth $a(\xi)$:

$$\begin{gathered}a(\xi) = \sum_{|\alpha|\leqslant n} c_\alpha(-i\xi)^\alpha + \sum_{|\beta|+|\gamma|\leqslant n} (-i\xi)^{\beta+\gamma}(\mathcal{F}_{\boldsymbol{G}}k_{\beta\gamma})(\xi), \quad a \in \mathfrak{M}^n_p(\widehat{\boldsymbol{G}}), \\ c_\alpha \in \mathbb{C}, \quad \xi \in \widehat{\boldsymbol{G}} := \boldsymbol{R}^n, \quad (-i\xi)^\alpha := (-i\xi_1)^{\alpha_1} \cdots (-i\xi_n)^{\alpha_n}.\end{gathered} \tag{2.4}$$

Here, we come to the celebrated Bessel potential spaces, but adapted to the underlying Lie group $\boldsymbol{G}$, and call such spaces Generic Bessel Potential Spaces (GBPS) $\mathbb{GH}^s_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$. Formally, they are defined as the classical Bessel potential spaces $\mathbb{H}^s_p(\mathbb{R}^n)$, but the Fourier transformation is different, associated with the Lie group $\boldsymbol{G}$: the norm in $\mathbb{GH}^s_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$ is written as follows (cf. [2]):

$$\begin{aligned}\|\psi \,|\, \mathbb{GH}^s_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})\| :&= \|W^0_{\boldsymbol{G},\langle\xi\rangle^s}\psi \,|\, \mathbb{L}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})\| \\ &= \|\mathcal{F}^{-1}_{\boldsymbol{G}}\langle\xi\rangle^s\mathcal{F}_{\boldsymbol{G}}\psi \,|\, \mathbb{L}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})\|, \quad \langle\xi\rangle^s := (1 + |\xi|^2)^s, \quad s \in \mathbb{R}.\end{aligned}$$

For an integer $s = m = 1, 2, \dots$, the space $\mathbb{GH}^m_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})$ is isomorphic to the Generic Sobolev Space (GSS) $\mathbb{GW}^m_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})$, consisting of functions with the finite norm

$$\|\varphi \,|\, \mathbb{GW}^m_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})\| := \Bigg[\sum_{|\alpha|\leqslant m} \|\mathfrak{D}^\alpha\varphi \,|\, \mathbb{L}_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})\|^p\Bigg]^{1/p}.$$

Lemma 2.2 (see [2]). *Let $1 < p < \infty$, $s, r \in \mathbb{R}$. The convolution operator $W^0_{\boldsymbol{G},a}$ with the symbol $a \in \mathfrak{M}^r_p(\widehat{\boldsymbol{G}})$ is bounded from the space $\mathbb{GH}^s_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$ to $\mathbb{GH}^{s-r}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$.*

Theorem 2.3 (Mikhlin–Hörmander–Lizorkin). *Let $1 < p < \infty$, $s, r \in \mathbb{R}$. If a function $a(\xi)$ satisfies the estimates*

$$|\xi^\alpha \partial^\alpha_\xi a(\xi)| \leqslant M_\alpha (1 + |\xi|^2)^{r/2} < \infty$$

for all $\alpha \in \mathbb{N}^n_0$, $|\alpha| \leqslant n/2 + 1$, $\alpha \leqslant 1$ (i.e., $\alpha_j \leqslant 1$, $j = 1, \dots, n$), then $a \in \mathfrak{M}^r_p(\widehat{\boldsymbol{G}})$ and

$$\|W^0_{G,a}\varphi \big| \mathbb{GH}^{s-r}_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})\| \leqslant C_p(a) \sum_{\substack{\alpha\leqslant 1\\ |\alpha|\leqslant n/2+1}} \sup_{\xi\in\mathbb{R}^n} |\xi^\alpha \partial^\alpha_\xi a(\xi)| \|\varphi \big| \mathbb{GH}^s_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})\|$$

for some constant $C_p(a) < \infty$.

Proof. Follows from the celebrated Mikhlin–Hörmander–Lizorkin theorem on $\mathbb{L}_p(\mathbb{R}^n)$-multipliers (see [17, Theorem7.9.5]) and is proved in [7, Proposition 5]. ☐

Let $W(\widehat{\boldsymbol{G}})$ be the Wiener algebra of matrix functions

$$W(\widehat{\boldsymbol{G}}) := \Big\{a(\xi) = c + (\mathcal{F}_G k)(\xi)] \;:\; k \in \mathbb{L}_1(\widehat{\boldsymbol{G}}, d\mu_{\boldsymbol{G}})\Big\}$$

and $W^r(\widehat{\boldsymbol{G}}) := \Big\{a(\xi) = \langle\xi\rangle^r a_0(\xi) \;:\; a_0 \in W(\widehat{\boldsymbol{G}})\Big\}$.

Theorem 2.4 (cf. [2]). *Let $1 < p < \infty$. An integro-differential convolution equation* (2.3)*, where $k_{\beta,\gamma}(x)$, $a(\xi)$ are $N \times N$ matrix-functions and $\varphi \in \mathbb{GH}^s_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$, $f \in \mathbb{GH}^{s-m}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$ are N-vectors, is Fredholm only if its symbol in* (2.4) *is elliptic*

$$\inf_{\xi\in\widehat{\boldsymbol{G}}} |\det\langle\xi\rangle^{-m} a(\xi)| > 0. \tag{2.5}$$

Moreover, if $a \in \boldsymbol{PC}^m_p(\widehat{\boldsymbol{G}})$, $a \in W(\widehat{\boldsymbol{G}})$, or satisfies conditions of Theorem 2.3, the ellipticity condition (2.5) *is sufficient for equation* (2.3) *to have a unique solution $\varphi = \boldsymbol{W}^0_{\boldsymbol{G},a^{-1}} f \in \mathbb{GH}^s_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$ for arbitrary $f \in \mathbb{GH}^{s-m}_p(\boldsymbol{G}, \boldsymbol{d}\mu_{\boldsymbol{G}})$.*

Remark 2.5. The foregoing Theorem 2.4 holds for the operator $W^0_{\boldsymbol{G},a} : \mathbb{H}^s_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}}) \to \mathbb{H}^{s-r}_p(\boldsymbol{G}, d\mu_{\boldsymbol{G}})$ for arbitrary $a \in \mathfrak{M}^r_p(\widetilde{\boldsymbol{G}})$, $r, s \in \mathbb{R}$ and $1 < p < \infty$.

Remark 2.6. Note that, due to the "Igari Paradox" (existence of a continuous elliptic symbol which is p-multiplier, but whose inverse is not p-multiplier for $p \neq 2$; cf. [18]), ellipticity of the symbol can only be a necessary condition for the Fredholm property in general.

Theorem 2.7 (cf. [7]). *For a convolution operator $W^0_{\boldsymbol{G},a}$ with an elliptic symbol $a \in \boldsymbol{PC}^m_p(\widehat{\boldsymbol{G}})$, the inverse operator $W^0_{\boldsymbol{G},a^{-1}}$ is a convolution operator, and its kernel $\mathcal{K}_a$ is the fundamental solution for $W^0_{\boldsymbol{G},a}$, i.e., $W^0_{\boldsymbol{G},a}\mathcal{K}_a = \delta$.*

Now we recall results about convolution integro-differential equations on submonoids from [8, 9], a generalization of Wiener–Hopf equations on the half-axes.

$\boldsymbol{M}$ is called a submonoid of the Lie group $\boldsymbol{G}$ if $\boldsymbol{M} \subset \boldsymbol{G}$, the binary operation from $\boldsymbol{G}$ maps $\boldsymbol{M}$ to itself, $\boldsymbol{M}$ contains the neutral element, but the inverses to $a \in \boldsymbol{M}$ do not belong to $\boldsymbol{M}$, but belongs to $\boldsymbol{G}$.

A convolution operator on the submonoid

$$W_{\boldsymbol{M},g}\varphi := (r_{\boldsymbol{M}} W^0_{\boldsymbol{G},g} \ell_{\boldsymbol{M}})\varphi, \quad g \in \mathfrak{M}_p(\mathbb{R}), \quad \operatorname{supp}\varphi \in \boldsymbol{M}, \tag{2.6}$$

represents the restriction of the convolution operator from the Lie group $\boldsymbol{G}$ to $\boldsymbol{M}$: $r_{\boldsymbol{M}}$ is the restriction of functions from $\boldsymbol{G}$ to $\boldsymbol{M}$ and $\ell_{\boldsymbol{M}}$ is the extension by 0 of functions from $\boldsymbol{M}$ to $\boldsymbol{G}$ (the right inverse to $r_{\boldsymbol{M}}$).

For $W_{\boldsymbol{M},g}$, the property (cf. [6])

$$W_{\mathcal{I},g}W_{\mathcal{I},h}=W_{\mathcal{I},gh},\quad g,h\in\mathfrak{M}_p(\mathbb{R}),\tag{2.7}$$

holds, in contrast to (2.1), if either $g(\xi)$ has an analytic extension $g(z)$ in the lower half-plane $\operatorname{Re} z<0$, or $h(\xi)$ has the analytic extension h(z) in the upper half plane $\operatorname{Re} z>0$.

To consider convolution integro-differential equations on a submonoid $\boldsymbol{M}=(0,1]$ of dimension 1,

$$\begin{aligned}\boldsymbol{W}_{\boldsymbol{M},a}\psi(x)&:=\sum_{i=0}^{n}c_i\mathcal{D}^i\varphi(x)+\sum_{j+\ell=0}^{m}\mathcal{D}^j\int\limits_{\boldsymbol{M}}k_{j\ell}(x\circ y^{-1})\mathcal{D}^\ell\psi(y)\boldsymbol{d}\mu_{\boldsymbol{M}}(y)=g(x),\\ a(\xi)&=\sum_{i=0}^{n}c_i(-i\xi)^i+\sum_{j+\ell=0}^{m}(-i\xi)^{j+\ell}(\mathcal{F}_{\boldsymbol{G}}k_{j\ell})(\xi),\quad a\in\mathfrak{M}^n_p(\widehat{\boldsymbol{G}}),\quad \xi\in\widehat{\boldsymbol{G}}:=\boldsymbol{R},\end{aligned}\tag{2.8}$$

in the Generic Bessel Potential Space setting, we need to introduce two different spaces on submonoid $\boldsymbol{M}$:

i. The space $\mathbb{GH}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$ consists of functions φ restricted from $\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})$, i.e., $\varphi=r_{\boldsymbol{M}}\psi$, $\psi\in\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})$. The norm in this space is defined as follows:

$$\begin{aligned}\|\varphi\,|\,\mathbb{GH}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})\|&=\inf_{\ell_{\boldsymbol{M}}\varphi\in\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})}\|\ell_{\boldsymbol{M}}\varphi\,|\,\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})\|\\&=\inf_{\ell_{\boldsymbol{M}}\varphi\in\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})}\|W^0_{\boldsymbol{G},\lambda^s_-}\ell_{\boldsymbol{M}}\varphi\,|\,\mathbb{L}_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})\|,\end{aligned}$$

where the infimum is taken over all extensions $\ell_{\boldsymbol{M}}\varphi\in\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})$.

ii. The space $\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$ represents the closure of the subset of functions from $\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})$ which have supports inside the interval $[0,1)$. Therefore, the embedding $\ell_{\boldsymbol{M}}\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})\subset\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})$ holds, and the norm is defined in a natural way:

$$\|\varphi\,|\,\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})\|=\|\ell_{\boldsymbol{M}}\varphi\,|\,\mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})\|_+=\|W^0_{\boldsymbol{G},\lambda^s_+}\ell_{\boldsymbol{M}}\varphi\,|\,\mathbb{L}_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})\|.$$

The next Lemma 2.8 is a natural consequence of Lemma 2.2.

Lemma 2.8 (see [2]). *Let $1<p<\infty$, $s,r\in\mathbb{R}$ and $\boldsymbol{M}\subset\boldsymbol{G}$ be a submonoid of a Lie group $\boldsymbol{G}$. A convolution operator on a submonoid $W_{\boldsymbol{M},a}$ with the symbol $a\in\mathfrak{M}^r_p(\widehat{\boldsymbol{G}})$ is bounded from the space $\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},\boldsymbol{d}\mu_{\boldsymbol{M}})$ to $\mathbb{GH}^{s-r}_p(\boldsymbol{M},\boldsymbol{d}\mu_{\boldsymbol{M}})$.*

The case of convolution equations on submonoid (cf. a particular case $r=m$ in (2.8))

$$W_{\boldsymbol{M},a}\psi=g,\quad\psi\in\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}}),\quad g\in\mathbb{GH}^{s-r}_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})\tag{2.9}$$

with a symbol $\alpha\in\mathfrak{M}^r_p(\mathbb{R})$ is much more delicate and complicated. We will consider the following three cases of 1-dimensional Lie groups:

- $\{\boldsymbol{G}.\circ\}=\{\mathbb{R},+\}$ and $\boldsymbol{M}=[0,\infty)$, the classical Wiener–Hopf equations (cf. [11, 12, 21, 24]);
- $\{\boldsymbol{G}.\circ\}=\{\mathbb{R}^+,\times\}$ and $\boldsymbol{M}=(0,1]$, the classical Mellin convolution equations (see Example 2.12 below and [4]);
- $\{\boldsymbol{G}.\circ\}=\left\{\mathcal{I}=(-1,1),a\circ b=\frac{a+b}{1+ab}\right\}$ and $\boldsymbol{M}=[0,1)$ (see Example 2.13 below and [6, 8, 9]).

In all three cases, the dual group of characters (where the symbols are defined) is homeomorphic to $\mathbb{R}$.

Cases of multidimensional groups are very complicated, even for the simplest cases of submonoides in $\mathbb{R}^n$, $n>1$.

To formulate the main result on Fredholm property of the convolution equation (2.9), we need to define the amended symbol. For this, we need first to lift the equation (2.9) to $\mathbb{L}_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$ space setting.

Corollary 2.9 (see [8, 9]). *The equation in* (2.9) *is lifted to the equivalent equation*

$$\begin{gathered} W_{\boldsymbol{M},a_{s,r}}\psi_0 = g_0, \quad \psi_0, g_0 \in \mathbb{L}_p(\boldsymbol{M}, d\mu_{\boldsymbol{M}}), \\ a_{s,r}(\xi) := \lambda_-^{-r}(\xi)\left(\frac{\xi - i}{\xi + i}\right)^s a(\xi) = \left(\frac{\xi - i}{\xi + i}\right)^s \frac{a(\xi)}{(\xi - i)^r}. \end{gathered} \tag{2.10}$$

The equations in (2.9) *and in* (2.10) *are Fredholm (are invertible) only simultaneously and, if they are Fredholm, they have the equal indices* $\mathbf{Ind}\, W_{\boldsymbol{M},a} = \mathbf{Ind}\, W_{\boldsymbol{M},a_{s,r}}$ *in the corresponding spaces.*

Moreover, the dimensions of kernels and co-kernels of these equations are equal.

Remark 2.10. Even if the symbol $a(\xi)$ of the equation in (2.9) is continuous, the symbol $a_{s,r}(\xi)$ of the lifted equation in (2.10) has discontinuity at infinity, provided s is not an integer:

$$a_{s,r}(-\infty) = a_{0,r}(-\infty), \quad a_{s,r}(+\infty) = a_{0,r}(+\infty)e^{2\pi si}.$$

To equation (2.9) with the symbol $a \in \boldsymbol{PC}_p^r(\widehat{\boldsymbol{G}})$, we associate the following amended symbol (see [8, 9]):

$$a_{s,r,p}(\xi,\eta) = \begin{cases} a_{s,r}(\xi) & \text{if} \quad \xi \neq c_1, c_2, \ldots, \\ a_{s,r}(c_k - 0)\frac{1+\cot\pi(i/p+\eta)}{2} + a_{s,r}(c_k + 0)\frac{1-\cot\pi(i/p+\eta)}{2}, \\ a_{s,r}(-\infty)\frac{1+\cot\pi(i/p+\eta)}{2} + a_{s,r}(\infty)\frac{1-\cot\pi(i/p+\eta)}{2}, & \xi,\eta \in \mathbb{R} \end{cases}$$

(see Fig. 1 and Fig. 2), where $a_{s,r}(\xi)$ is the lifted symbol in (2.10), $c_1, c_2, \ldots$ are all points of discontinuities of $a_{s,r}(\xi)$. It is clear that $a_{s,r}(-\infty) = a_{0,r}(-\infty)$ and $a_{s,r}(\infty) = e^{2\pi si}a_{0,r}(\infty)$.

We say that the amended symbol $a_p^s(\xi,\eta)$ is elliptic if

$$\inf_{\xi \in \widehat{\boldsymbol{G}}} |a_{s,r,p}(\xi,\eta)| > 0. \tag{2.11}$$

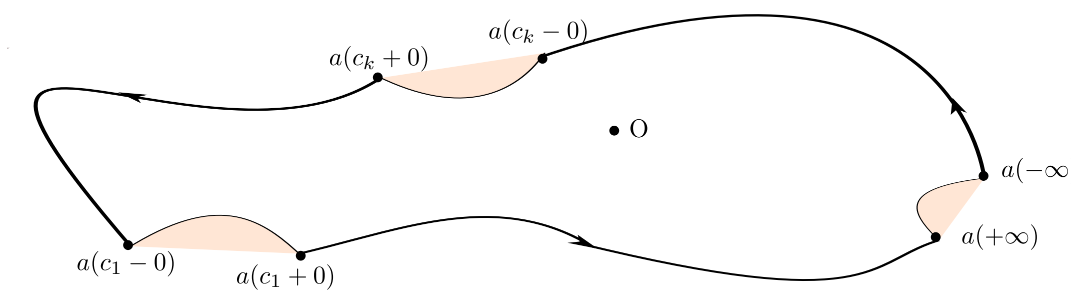

Fig. 1: Image of the amended symbol

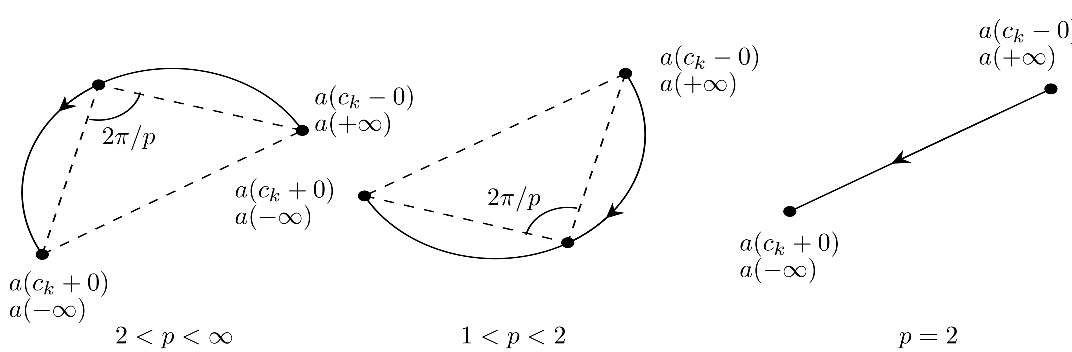

Fig. 2: Arc condition at jumps

If the amended symbol is elliptic, we can define the integer index of the amended symbol

$$\text{ind } a_{s,r,p} = \frac{1}{2\pi}\left[\arg a_{s,r,p}(\xi,\eta)\right]_{(\xi,\eta)\in\mathbb{R}^2}$$

as follows: consider the increment of the argument $\frac{1}{2\pi}\arg a_{s,r,p}(\xi,\eta)$ when ξ ranges through $\mathbb{R}=(-\infty,\infty)$, from $-\infty$ to ∞, between the points of discontinuities $c_1, c_2, \dots$ and $c_0=\infty$ of the function; at the discontinuity point c_j the argument η ranges through $\mathbb{R}$, connecting $a_{s,r}(c_j-0)$ to $a_{s,r}(c_j+0)$, $j=1,2,\dots$, or $a_{s,r}(c_0-0)=a_{s,r}(\infty)$ to $a_{s,r}(c_0+0)=a_{s,r}(-\infty)$ with arcs defined by $\cot\pi(i/p+\eta)$ (the line segment when $p=2$) (see Fig. 1 and Fig. 2).

Theorem 2.11 (see [8, 9]). *Let $1<p<\infty$, $s\in\mathbb{R}$. A convolution equation in the Generic Bessel Potential Space setting* (2.9) *with the matrix symbol $a\in\mathfrak{M}^r_p(\widehat{\boldsymbol{G}})$ is Fredholm only if the symbol is elliptic* $\inf_{\xi\in\widehat{\boldsymbol{G}}}|\det a(\xi)|>0$.

If $a\in\boldsymbol{PC}^r_p(\widehat{\boldsymbol{G}})$ or $a\in W^r(\widehat{\boldsymbol{G}})$ is scalar symbol, the ellipticity of the amended symbol (2.11) *is necessary and sufficient for equation* (2.9) *to be Fredholm.*

If equation (2.9) *is Fredholm, its index is*

$$\textbf{Ind } W_{\boldsymbol{M},a} = -\text{ind } a_{s,r,p}$$

and:

i. *For* ind $a_{s,r,p}=0$, *the equation has a unique solution $u\in\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$ for all $w\in\mathbb{GH}^{s-r}_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$;*

ii. *For $\kappa=-\text{ind } a_{s,r,p}>0$, the equation has a solution for all $w\in\mathbb{GH}^{s-r}_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$, but the solution is not unique and the homogeneous equation $w=0$ has exactly κ linearly independent solutions $u_1,\dots,u_\kappa\in\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$;*

iii. *For $\kappa=\text{ind } a_{s,r,p}>0$, the equation has a unique solution $u\in\widetilde{\mathbb{GH}}^s_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$ for all those $w\in\mathbb{GH}^{s-r}_p(\boldsymbol{M},d\mu_{\boldsymbol{M}})$, which are orthogonal to the solutions of the dual homogeneous equation $W_{\boldsymbol{M},\overline{a}}v=0$ in the dual space $v_1,\dots v_\kappa\in\mathbb{GH}^{-s}_{p'}(\widetilde{\boldsymbol{G}},d\mu_{\boldsymbol{G}})$, $p'=p/(p-1)$.*

Example 2.12 (cf. [7]). The direct product of positive half-axes $\mathbb{R}^n_+=(0,\infty)^n$ is a Lie group with the group operation $x\circ y=(x_1y_1,\dots,x_ny_n)$, with the dual $\widehat{\mathbb{R}^n_+}=\mathbb{R}^n$ and the Haar measure $\frac{\boldsymbol{d}x_1}{x_1}\cdots\frac{\boldsymbol{d}x_n}{x_n}$. The group Fourier transformation on $\mathbb{R}^n_+$ coincides with the Mellin transformation (see [6]), and the generic differential operators and Generic Laplacian in $\mathbb{R}^n_+$ are

$$\Delta_{\mathbb{R}^n_+}=\mathcal{D}^2_1+\cdots+\mathcal{D}^2_n,\quad \mathfrak{D}_k=x_k\partial_k,\quad k=1,2,\dots,n.$$

The fundamental solution of the Generic Laplacian is

$$\mathcal{K}_{\Delta_{\mathbb{R}^n_+}}(x):=\begin{cases}\frac{1}{4\pi}\ln(\ln^2 x_1+\ln^2 x_2) & \text{if } n=2,\\ \frac{\Gamma(n/2)}{2(2-n)\pi^{n/2}}(\ln^2 x_1+\cdots+\ln^2 x_n)^{\frac{2-n}{2}} & \text{if } n>2.\end{cases}$$

Example 2.13 (cf. [7]). The direct product of intervals $I^n=(-1,1)^n$ is a Lie group with the group operation (cf. [6])

$$x\circ y=\left(\frac{x_1+y_1}{1+x_1y_1},\dots,\frac{x_1+y_1}{1+x_1y_1}\right),\quad x,y\in I^n,$$

with the Haar measure $\frac{\boldsymbol{d}x_1}{1-x_1^2}\cdots\frac{\boldsymbol{d}x_n}{1-x_n^2}$, and the generic differential operators and Generic Laplacian are

$$\Delta_{\mathbb{R}^n_+}=\mathcal{D}^2_1+\cdots+\mathcal{D}^2_n\quad \mathcal{D}_k=(1-x_k^2)\partial_k,\quad k=1,2,\dots,n.$$

Then the fundamental solution of the Generic Laplacian is

$$\mathcal{K}_{\Delta_{\mathbb{R}^n_+}}(x):=\begin{cases}\frac{1}{4\pi}\ln\frac{1}{4}\left[\ln^2\frac{1+x_1}{1-x_1}+\ln^2\frac{1+x_2}{1-x_2}\right] & \text{if } n=2,\\ \frac{2^{n/2-2}\Gamma(n/2)}{(2-n)\pi^{n/2}}\left[\ln^2\frac{1+x_1}{1-x_1}+\cdots+\ln^2\frac{1+x_n}{1-x_n}\right]^{\frac{2-n}{2}} & \text{if } n>2.\end{cases}$$

3 Hadamard type fractional derivatives on Lie groups

As was shown in Section 2, the generic differential operators $\{\mathfrak{D}_k\}_{k=1}^n$ on a Lie group $\boldsymbol{G}$ are convolution operators (cf. (2.2)),

Let us define derivatives of non-integer positive order on the Lie group as follows:

$$\mathfrak{D}^\alpha\varphi(x) := (W^0_{(-i\xi)^\alpha}\varphi)(x) = \int_0^\infty k_\alpha(x\circ y^{-1})\varphi(y)\,d\mu_{\boldsymbol{G}}y, \quad x\in\boldsymbol{G}, \quad \alpha>0. \tag{3.1}$$

Due to the properties of convolution operators $W^0_{\boldsymbol{G},a}$ (cf. (2.1)–(2.2)), the Hadamard type fractional derivatives have the following properties:

$$\begin{aligned}\mathfrak{D}^\alpha\ &:\ \mathbb{GH}^s_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})\to\mathbb{GH}^{s-\alpha}_p(\boldsymbol{G},d\mu_{\boldsymbol{G}})\ \text{ is bounded for all } s\in\mathbb{R},\ \alpha>0,\\ &(\mathfrak{D}^\alpha\mathfrak{D}^\gamma\varphi)(x) = (\mathfrak{D}^{\alpha+\gamma}\varphi)(x), \quad \alpha,\gamma>0, \quad x\in\boldsymbol{G},\\ &(\mathcal{F}_{\boldsymbol{G}}\mathfrak{D}^\alpha\varphi)(\xi) = (-i\xi)^\alpha(\mathcal{F}_{\boldsymbol{G}}\varphi)(\xi), \quad \xi\in\mathbb{R}.\end{aligned} \tag{3.2}$$

3.1 The modified Hadamard type fractional derivatives on the Lie group $\{\mathbb{R}^+,\cdot\}$

As was mentioned in Section 2, the operator $\mathfrak{D} = xd/dx$ is the generic differential operator on the Lie group $\{\mathbb{R}^+,\cdot\}$ (i.e., is a convolution operator on this group):

$$(\mathfrak{D}_-\varphi)(x) := x\frac{d}{dx}\varphi(x) = (\mathfrak{M}^0_{i\xi-\beta}\varphi)(x), \quad x\in\mathbb{R}, \tag{3.3a}$$

$$(\mathfrak{D}_+\varphi)(x) := \left[x\frac{d}{dx}+2\beta I\right]\varphi(x) = (\mathfrak{M}^0_{i\xi+\beta}\varphi)(x), \quad x\in\mathbb{R}, \tag{3.3b}$$

$$(\mathfrak{D}_{-,I}\psi)(t) := r_I\left[t\frac{d}{dt}\right]\psi(t) = (\mathfrak{M}_{i\xi-\beta}\psi)(t), \quad t\in I=(0,1), \tag{3.3c}$$

$$(\mathfrak{D}_{+,I}\psi)(t) := r_I\left[t\frac{d}{dt}+\beta I\right]\psi(t) = (\mathfrak{M}_{i\xi+\beta}\psi)(t), \quad t\in I=(0,1), \tag{3.3d}$$

where r_I is the restriction operator to the interval $I=(0,1)$, $\mathcal{M}_\beta$ and $\mathcal{M}^{-1}_\beta$ are the Mellin transformation (the Fourier transform on the group $\{\mathbb{R}^+,\cdot\}$) and its inverse:

$$(\mathcal{M}_\beta\varphi)(\xi) := \int_0^\infty y^{\beta-i\xi}\varphi(y)\,\frac{dy}{y}, \quad \xi\in\mathbb{R}, \tag{3.3e}$$

$$(\mathcal{M}^{-1}_\beta f)(x) = \frac{1}{2\pi}\int_{-\infty}^\infty x^{i\xi-\beta}f(\xi)\,d\xi, \quad x\in\mathbb{R}^+, \quad 0<\beta<1, \tag{3.3f}$$

while $\mathfrak{M}^0_g$ is the Mellin convolution operator with the symbol g:

$$\mathfrak{M}^0_g\varphi(x) := \mathcal{M}^{-1}_\beta g\mathcal{M}_\beta\varphi(x) = \int_0^\infty k\Big(\frac{x}{y}\Big)\varphi(y)\,\frac{dy}{y}, \quad g(\xi)=\mathcal{M}_\beta k(\xi), \quad \xi\in\mathbb{R}.$$

Here, $k(t)$ is the distributional Schwartz kernel of the Mallin convolution operator $\mathfrak{M}^0_g$.

The modified Hadamard type derivatives of fractional order $\alpha\in\mathbb{R}$ are defined as follows (see [19, p. 111]):

$$(\mathfrak{D}^\alpha_\pm\varphi)(x) = (\mathfrak{M}^0_{(i\xi\pm\beta)^\alpha}\varphi)(x), \quad x\in\mathbb{R}^+ := (0,\infty), \tag{3.4a}$$

$$(\mathfrak{D}^\alpha_{\pm,I}f)(t) = (\mathfrak{M}_{(i\xi\pm\beta)^\alpha}\varphi)(t), \quad t\in I=(0,1). \tag{3.4b}$$

In contrast to the fractional differential operators $\mathfrak{D}^\alpha_k$, defined in (3.1), the operators $\mathfrak{D}^\alpha_\pm$ and $\mathfrak{D}^\alpha_{\pm,I}$, defined in (3.4a) and (3.4b), have the advantage that they are also defined for negative powers $\alpha<0$.

Theorem 3.1. *The modified Hadamard type derivatives* $\mathfrak{D}^\alpha_\pm$ *and* $\mathfrak{D}^\alpha_{\pm,I}$ *of fractional order* $\alpha\in\mathbb{R}$*, defined in* (3.4a) *and* (3.4b)*, have the following forms:*

$$(\mathfrak{D}^\alpha_-\varphi)(x)=\frac{e^{i\pi\alpha}}{\Gamma(-\alpha)}\int\limits_x^\infty\left(\ln\frac{y}{x}\right)^{-\alpha-1}\varphi(y)\,\frac{dy}{y},\quad x\in\mathbb{R}^+=(0,\infty),\tag{3.5a}$$

$$(\mathfrak{D}^\alpha_{-,I}\psi)(t)=\frac{e^{i\pi\alpha}}{\Gamma(-\alpha)}\int\limits_t^1\left(\ln\frac{\tau}{t}\right)^{-\alpha-1}\psi(\tau)\,\frac{d\tau}{\tau},\quad t\in I=(0,1],\tag{3.5b}$$

$$(\mathfrak{D}^\alpha_+\varphi)(x)=\frac{1}{\Gamma(-\alpha)}\int\limits_0^x\left(\ln\frac{x}{y}\right)^{-\alpha-1}\varphi(y)\,\frac{dy}{y},\quad x\in\mathbb{R}^+=(0,\infty),\tag{3.5c}$$

$$(\mathfrak{D}^\alpha_{+,I}\psi)(t)=\frac{1}{\Gamma(-\alpha)}\int\limits_0^t\left(\ln\frac{t}{\tau}\right)^{-\alpha-1}\psi(\tau)\,\frac{d\tau}{\tau},\quad t\in I:=(0,1),\tag{3.5d}$$

and map the following spaces:

$$\mathfrak{D}^\alpha_\pm\ :\ \mathbb{GH}^s_p(\mathbb{R}^+,dt/t)\to\mathbb{GH}^{s-\alpha}_p(\mathbb{R}^+,dt/t),\tag{3.5e}$$

$$\mathfrak{D}^\alpha_{\pm,I}\ :\ \widetilde{\mathbb{GH}}^s_p(I,dt/t)\to\mathbb{GH}^{s-\alpha}_p(I,dt/t)\quad\text{for all } s\in\mathbb{R},\ \alpha\in\mathbb{R}.\tag{3.5f}$$

Proof. Let first $\alpha<-1$.

To find the distributional kernel $k^\alpha_-(t)$ of $\mathfrak{D}^\alpha_-$ (and of $\mathfrak{D}^\alpha_{-,I}$), we proceed as follows:

$$k^\alpha_-(t)=(\mathcal{M}^{-1}_\beta(i\xi-\beta)^\alpha)(t)=\frac{1}{2\pi}\int\limits_{-\infty}^\infty t^{i\xi-\beta}(i\xi-\beta)^\alpha d\xi=\frac{e^{i\pi\alpha}t^{-\beta}}{2\pi}\int\limits_{-\infty}^\infty t^{i\xi}(\beta-i\xi)^\alpha d\xi$$
$$=\frac{e^{i\pi\alpha}t^{-\beta}}{2\pi}R_*\int\limits_{-\infty}^\infty e^{i\xi x}(\beta-i\xi)^\alpha d\xi,$$

where $R_*\varphi(x):=\varphi(\ln t)$. To the obtained integral we apply Formulae [13, 3.382.7]:

$$\begin{aligned}k^\alpha_-(t)&=\frac{e^{i\pi\alpha}t^{-\beta}}{\Gamma(-\alpha)}R_*\left[\theta_-(t)(-x)^{-\alpha-1}e^{\beta x}\right](t)\\&=\theta_I(t)\frac{e^{i\pi\alpha}(-\ln t)^{-\alpha-1}}{\Gamma(-\alpha)},\quad t\in\mathbb{R}^+,\quad\alpha<-1,\end{aligned}\tag{3.5g}$$

where $\theta_-(x)$ and $\theta_I(t)$ are, respectively, the characteristic functions of the half-axes $\mathbb{R}^-=(-\infty,0)$ and of the interval $I=(0,1)$.

From (3.7) it follows (3.5a) and (3.5b) for $\alpha<-1$.

To find the distributional kernel $k^\alpha_+(t)$ of $\mathfrak{D}^\alpha_+$ (and of $\mathfrak{D}^\alpha_{+,I}$), we proceed as follows:

$$k^\alpha_+(t)=(\mathcal{M}^{-1}_\beta(i\xi+\beta)^\alpha)(t)=\frac{1}{2\pi}\int\limits_{-\infty}^\infty t^{i\xi+\beta}(i\xi+\beta)^\alpha d\xi=\frac{t^\beta}{2\pi}\int\limits_{-\infty}^\infty t^{i\xi}(\beta+i\xi)^\alpha d\xi$$
$$=\frac{t^\beta}{2\pi}R_*\int\limits_{-\infty}^\infty e^{i\xi x}(\beta+i\xi)^\alpha d\xi,$$

where $R_*\varphi(x):=\varphi(\ln t)$. To the obtained integral we apply Formulae [13, 3.382.6]:

$$k^\alpha_+(t)=\frac{t^\beta}{\Gamma(-\alpha)}R_*\left[\theta_+(x)x^{-\alpha-1}e^{-\beta x}\right](t)=\theta_{(1,\infty)}(t)\frac{(\ln t)^{-\alpha-1}}{\Gamma(-\alpha)},\quad t\in\mathbb{R}^+,\quad\alpha<-1,\tag{3.5h}$$

where $\theta_+(x)$ and $\theta_{(1,\infty)}(t)$ are, respectively, the characteristic functions of the half axes $\mathbb{R}^+ = (0,\infty)$ and of the infinite interval $(1,\infty)$.

From (3.5h) it follows (3.5c) and (3.5d) for $\alpha < -1$.

If $\alpha \geqslant -1$, we choose the integer $m = 1, 2, \dots$ such that $-2 \leqslant \alpha - m < 1$. Since $\mathfrak{D}^\alpha_- = \mathfrak{D}^m_- \mathfrak{D}^{\alpha-m}_-$, we get

$$\begin{aligned}
(\mathfrak{D}^\alpha_-\varphi)(x) &= \frac{e^{i\pi(\alpha+m)}}{\Gamma(-\alpha+m)}\mathfrak{D}^m_- \int\limits_x^\infty \left(\ln\frac{y}{x}\right)^{-\alpha+m-1}\varphi(y)\,\frac{dy}{y}\\
&= \frac{e^{i\pi(\alpha+m)}}{\Gamma(-\alpha+m)}\mathfrak{D}^{m-1}_- \left[x\left(\ln\frac{y}{x}\right)^{-\alpha+m-1}\varphi(y)\,\frac{dy}{y}\right]_{y=x}\\
&\quad + \frac{e^{i\pi\alpha}(-1)^m}{\Gamma(-\alpha+m-1)(-\alpha+m-1)}\mathfrak{D}^{m-1}_- \int\limits_x^\infty x\frac{d}{dx}\left(\ln\frac{y}{x}\right)^{-\alpha+m-1}\varphi(y)\,\frac{dy}{y}\\
&= \frac{e^{i\pi\alpha}(-1)^{m-1}}{\Gamma(-\alpha+m-1)}\mathfrak{D}^{m-1}_- \int\limits_x^\infty \left(\ln\frac{y}{x}\right)^{-\alpha+m-2}\varphi(y)\,\frac{dy}{y} = \cdots =\\
&= \frac{e^{i\pi\alpha}}{\Gamma(-\alpha)} \int\limits_x^\infty \left(\ln\frac{y}{x}\right)^{-\alpha-1}\varphi(y)\,\frac{dy}{y},
\end{aligned}$$

and (3.5a), (3.5b) follow also for $\alpha \geqslant -1$.

Formulae (3.5c) and (3.5d) for $\alpha \geqslant -1$ are proved similarly.

The mapping properties (3.5e) and (3.5f) follow from Lemma 2.2 and Lemma 2.8. □

Lemma 3.2. *The modified Hadamard fractional derivatives are all invertible:*

$$\mathfrak{D}^\alpha_\pm \mathfrak{D}^{-\alpha}_\pm \varphi(x) = \mathfrak{D}^{-\alpha}_\pm \mathfrak{D}^\alpha_\pm \varphi(x) = \varphi(x), \quad x \in \mathbb{R}^+, \tag{3.6a}$$

$$\mathfrak{D}^\alpha_{\pm,I} \mathfrak{D}^{-\alpha}_{\pm,I} \psi(x) = \mathfrak{D}^{-\alpha}_{\pm,I} \mathfrak{D}^\alpha_{\pm,I} \psi(x) = \psi(x), \quad x \in I = (0,1], \tag{3.6b}$$

for all $\alpha \in \mathbb{R}$.

The functions (see (3.7), (3.5h)*)*

$$k^\alpha_-(t) = \theta_I(t)\frac{e^{i\pi\alpha}(-\ln t)^{-\alpha-1}}{\Gamma(-\alpha)}, \quad k^\alpha_+(t) = \theta_{(1,\infty)}(t)\frac{(\ln t)^{-\alpha-1}}{\Gamma(-\alpha)}, \quad t \in \mathbb{R}^+, \quad \alpha < -1, \tag{3.7}$$

are fundamental solutions of $\mathfrak{D}^\alpha_-$ *and* $\mathfrak{D}^\alpha_+$*, respectively:*

$$\mathfrak{D}^\alpha_\pm k^{-\alpha}_\pm(x) = \delta(x) \quad \text{for all } \alpha \in \mathbb{R},\ x \in \mathbb{R}^+.$$

Proof. Equalities (3.6a) follow from (2.1), while equalities (3.6b) follow from (2.9), since the symbols $(i\xi - \beta)^{\pm\alpha}$ of the operators $\mathfrak{D}^{\pm\alpha}_-$ have the analytic extension in the upper half-plane $\operatorname{Im}\xi > 0$, and the symbols $(i\xi + \beta)^{\pm\alpha}$ of the (operators $\mathfrak{D}^{\pm\alpha}_+$ have the analytic extension in the lower half plane $\operatorname{Im}\xi < 0$.

Since $k^{-\alpha}_\pm$ are the kernels of the inverse operators $\mathfrak{D}^{-\alpha}_\pm$ to $\mathfrak{D}^\alpha_\pm$, the concluding part of the lemma on fundamental solutions follows directly from Theorem 2.4. □

Due to (2.1) and (2.6), the modified Hadamard type fractional derivatives have the following properties:

$$\begin{aligned}
&(\mathfrak{D}^\alpha_\pm \mathfrak{D}^\gamma_\pm \varphi)(x) = (\mathfrak{D}^{\alpha+\gamma}_\pm \varphi)(x), \quad (\mathfrak{D}^\alpha_{\pm,I} \mathfrak{D}^\gamma_{\pm,I} \psi)(t) = (\mathfrak{D}^{\alpha+\gamma}_{\pm,I} \psi)(t), \quad \mathfrak{D}^0_\pm = \mathfrak{D}^0_{\pm,I} = I,\\
&(\mathcal{M}_\beta \mathfrak{D}^\alpha_\pm \varphi)(\xi) = (i\xi \pm \beta)^\alpha (\mathcal{M}_\beta \varphi)(\xi), \quad \alpha, \gamma \in \mathbb{R}, \quad \xi \in \mathbb{R}, \quad x \in \mathbb{R}^+, \quad t \in I.
\end{aligned}$$

For the following integro-differential (pseudodifferential, convolution) equations with the modified Hadamard type fractional differential operators

$$\boldsymbol{W}_{\pm,I,a}\varphi(t) := \sum_{i=0}^{m} c_i \mathcal{D}^{\alpha_i}_{\pm,I}\varphi(t) + \sum_{j+\ell=0}^{n} \mathcal{D}^{\omega_j}_{\pm,I} \int_0^1 k_{j\ell}\left(\frac{t-\tau}{1-t\tau}\right) \mathcal{D}^{\gamma_\ell}_{\pm,I}\varphi(\tau)\frac{d\tau}{\tau} = f(t), \tag{3.8}$$

$$\begin{aligned} &c_j \in \mathbb{C}, \quad \alpha_j, \beta_j, \gamma_j \in \mathbb{R}, \quad k_{j\ell} \in \mathbb{L}_1(\boldsymbol{G}, dt/t), \quad j, \ell = 0, \dots, n, \\ &\varphi \in \widetilde{\mathbb{GH}}^s_p(\boldsymbol{M}, dt/t, \quad f \in \mathbb{GH}^{s-r}_p(\boldsymbol{M}, dt/t) \end{aligned} \tag{3.9}$$

with symbols of power growth at the infinity

$$\begin{aligned} &a_\pm(\xi) = \sum_{i=0}^{n} c_i (i\xi \pm \beta)^{\alpha_i} + \sum_{\omega_j+\gamma_\ell=0}^{n} (i\xi \pm \beta)^{\omega_j+\gamma_\ell} (\mathcal{F}_{\boldsymbol{G}} k_{\beta,\gamma})(\xi), \quad a \in \mathfrak{M}^r_p(\widehat{\boldsymbol{G}}), \qquad (3.10) \\ &\xi \in \widehat{\boldsymbol{G}} = \mathbb{R}\,, \quad r := \max\{\alpha_1, \dots, \alpha_n, \omega_1 + \gamma_1, \dots, \omega_n + \gamma_n\}, \end{aligned}$$

we have the following corollary of Theorem 2.7.

Corollary 3.3. *For equations* (3.8) *in setting* (3.9) *and with the symbols* $a_\pm(\xi)$ *in* (3.10) *of order* r, *all conclusions of Theorem 2.7 hold.*

3.2 The Hadamard type fractional derivatives on the Lie group $\{\mathcal{I} = (-1, 1), \circ\}$

The interval $\mathcal{I} = (-1, 1)$ is a Lie group with the group operation $x \circ y := \frac{x+y}{1+xy}$ (cf. (2.7)) and the generic differential operator (cf. [6]):

$$(\mathfrak{D}\varphi)(x) := (1-x^2)\frac{d\varphi(x)}{dx} = (W^0_{\mathcal{I},-i\xi}\varphi)(x) = (\mathcal{F}^{-1}_{\mathcal{I}}(-i\xi)\mathcal{F}_{\mathcal{I}}\varphi)(x), \quad x \in \mathcal{I}\,.$$

Consider the restriction of $\mathfrak{D}$ to the unit subinterval $I = (0, 1)$

$$(\mathfrak{D}_I\psi)(t) := r_I\left[(1-t^2)\frac{d\psi(t)}{dt}\right] = (W_{\mathcal{I},-i\xi}\psi)(t) = (r_I W^0_{\mathcal{I},-i\xi}\psi)(t), \quad t \in I,$$

where r_I is the restriction operator to I. Fourier transformation on the group $\{\mathcal{I}, \circ\}$ and its inverse are

$$(\mathcal{F}_{\mathcal{I}}\varphi)(\xi) := \int_{-1}^{1} \left(\frac{1+y}{1-y}\right)^{i\xi} \varphi(y)\frac{dy}{1-y^2}, \quad \xi \in \mathbb{R},$$

$$(\mathcal{F}^{-1}_{\mathcal{I}}\psi)(x) := \frac{1}{\pi^m}\int_{-\infty}^{\infty} \left(\frac{1+x}{1-x}\right)^{-i\xi} \psi(\xi/2)\,\boldsymbol{d}\xi, \quad x \in \mathbb{R}^+, \quad 0 < \beta < 1,$$

while $W^0_{\mathcal{I},g}$ is the convolution operator with the symbol $g(\xi)$:

$$\begin{aligned} W^0_{\mathcal{I},g}\varphi(t) := \mathcal{F}^{-1}_{\mathcal{I}}\, g\mathcal{F}_{\mathcal{I}}\varphi(t) = \int_{-1}^{1} k\left(\frac{x-y}{1-xy}\right)\varphi(y)\,\frac{dy}{1-y^2}, \\ g(\xi) = \mathcal{F}_{\mathcal{I}} k(\xi), \quad \xi \in \mathbb{R}. \end{aligned}$$

Here, $k(t)$ is the distributional Schwartz kernel of the convolution operator $W^0_{\mathcal{I},g}$.

The modified Hadamard type derivatives of fractional order $\alpha > 0$ are defined as follows:

$$(\mathfrak{D}^\alpha_{\mathcal{I}}\varphi)(x) = (W^0_{\mathcal{I},(-i\xi)^\alpha}\varphi)(x) = \int_{-1}^{1} k_\alpha\left(\frac{x-y}{1-xy}\right)\varphi(y)\,\frac{dy}{1-y^2},$$

$$(\mathfrak{D}^\alpha_{\mathcal{I},I}\psi)(t) = (W^0_{\mathcal{I},(-i\xi)^\alpha}\psi)(t) = \int_0^1 k_\alpha\left(\frac{t-\tau}{1-t\tau}\right)\psi(\tau)\,\frac{d\tau}{1-\tau^2}, \quad x \in \mathcal{I} = (-1,1), \quad t \in I = (0,1).$$

Due to (2.1) and (2.6), the modified Hadamard type fractional derivatives have the following properties:

$$(\mathfrak{D}^{\alpha}_{\mathcal{I}}\mathfrak{D}^{\gamma}_{\mathcal{I}}\varphi)(x) = (\mathfrak{D}^{\alpha+\gamma}_{\mathcal{I}}\varphi)(x), \quad (\mathcal{F}_{\mathcal{I}}\mathfrak{D}^{\alpha}_{\mathcal{I}}\varphi)(\xi) = (i\xi-\beta)^{\alpha}(\mathcal{F}_{\mathcal{I}}\varphi)(\xi), \quad \alpha,\gamma\in\mathbb{R}^+, \quad \xi\in\mathbb{R}, \quad x\in\mathbb{R}^+.$$

Let us conclude this section by the following definitions.

Definition 3.4. The Kilbas–Saigo function $E_{\alpha,m,l}(z)$ is defined in [19, Remark 5.1] as follows:

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k, \quad c_0 = 1, \quad c_k = \Pi_{j=0}^{k-1}\frac{\Gamma(\alpha(jm+l)+1)}{\Gamma(\alpha(jm+l+1)+1)}, \quad k\geq 1,$$

where $m = 1,2,3,\dots$, $l = -1,0,1,\dots$, $\alpha\in(0,1)$.

Theorem 3.5 (see [1, Theorem 2, Remark 4]). *Let $0<\alpha\leqslant 1$, $m>0$, $x\geqslant 0$. The function $E_{\alpha,m,\ell}(z)$, $z\in\mathbb{C}$, is an entire function, $E_{\alpha,m,\ell}(-x)$, $x>0$, is uniformly bounded and*

$$E_{\alpha,m,\ell}(-x) = \mathcal{O}\left(x^{-1}\right) \quad \text{as} \quad x\to\infty.$$

Definition 3.6. The Mittag–Leffler function $E_{\alpha,\beta}(z)$ is defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta\in\mathbb{C}, \quad \operatorname{Re}\alpha>0.$$

Lemma 3.7 (see [19, pp. 42, 43]). *The function $E_{\alpha,\beta}(z)$, $z\in\mathbb{C}$, is entire and has the derivatives*

$$\left(\frac{d}{dz}\right)^n E_{\alpha,\beta}(z) = n!E^{n+1}_{\alpha,\beta+\alpha n}(z), \quad n=1,2,\dots.$$

Moreover, $E_{\alpha,\beta}(-x)$ and its derivatives are uniformly bounded for $0<\alpha<2$, $x>0$, and have the following asymptotics:

$$E_{\alpha,\beta}(-x) = \mathcal{O}\left(x^{-1}\right) \quad \text{as} \quad x\to\infty,$$
$$\left(\frac{d}{dz}\right)^n E_{\alpha,\beta}(z) = \mathcal{O}\left(x^{-n-1}\right) \quad \text{as} \quad x\to\infty.$$

4 Time-fractional diffusion equations

4.1 Cauchy problem for $\mathfrak{D}^{\alpha}_{+,t} - \lambda(t-a)^{\gamma}\Delta_x$ on the domain $\mathbb{R}^+_a\times\mathbb{R}^N$

Let Δ_x be the Laplace operator on $\mathbb{R}^N$ and let $\mathfrak{D}^{\alpha}_{+,t}$ be the modified Hadamard type fractional derivative of order $\alpha\in\mathbb{R}$ on the infinite interval $\mathbb{R}^+_a := (a,\infty)$ for some $a>0$, defined in (3.3a) and (3.5c).

Theorem 4.1. *Let $1<\alpha<2$, $\gamma>-\{\alpha\}$, $\lambda\in\mathbb{R}$, $1<p<\infty$, $s\in\mathbb{R}$. Then the Cauchy problem for the time-fractional diffusion equation*

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}u(t,x) - \lambda(t-a)^{\gamma}\Delta_x u(t,x) = 0\,, \\ (\mathfrak{D}^{\alpha-1}_{+,t}u)(0,x) = g(x)\,, \quad (\mathfrak{D}^{\alpha-2}_{+,t}u)(0,x) = h(x)\,, \quad t\in\mathbb{R}^+_a, \quad x\in\mathbb{R}^N, \end{cases} \tag{4.1}$$
$$\left(\log\frac{t}{a}\right)^{2-\alpha} u \in \bigcap C(\mathbb{R}^+_a;\mathbb{H}^s(\mathbb{R}^N)), \quad g,h\in\mathbb{H}^s(\mathbb{R}^N),$$

has a unique solution of the form

$$u(t,x) = \frac{\left(\log\frac{t}{a}\right)^{\alpha-1}}{(2\pi)^N\Gamma(\alpha)}\int_{\mathbb{R}^N} e^{-ix\xi}\widetilde{g}(\xi)E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-1}{\alpha}}\left(-\lambda|\xi|^2\left(\log\frac{t}{a}\right)^{\gamma+\alpha}\right)d\xi$$

$$+\frac{\left(\log\frac{t}{a}\right)^{\alpha-2}}{(2\pi)^N\Gamma(\alpha-1)}\int\limits_{\mathbb{R}^N}e^{-ix\xi}\widetilde{h}(\xi)E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-1}{\alpha}}\left(-\lambda|\xi|^2\left(\log\frac{t}{a}\right)^{\gamma+\alpha}\right)d\xi$$

$$=\frac{\left(\log\frac{t}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}W^0_{\mathbb{R}^N,\mathcal{E}_1(t,\cdot)}g(t,x)+\frac{\left(\log\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha-1)}W^0_{\mathbb{R}^N,\mathcal{E}_2(t,\cdot)}h(t,x), \tag{4.2}$$

where $E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-1}{\alpha}}(z)$ *is the Kilbas–Saigo function (see Definition 3.4) and*

$$\mathcal{E}_k(t,\xi):=E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-k}{\alpha}}\left(-\lambda|\xi|^2\left(\log\frac{t}{a}\right)^{\gamma+\alpha}\right),\quad k=1,2,\quad \widetilde{g}(\xi)=\int\limits_{\mathbb{R}^N}e^{is\xi}g(s)ds. \tag{4.3}$$

Proof. Applying the Fourier transformation in x to problem (4.1) we obtain the following equivalent Cauchy problem with the parameter:

$$\begin{cases}\mathfrak{D}^\alpha_{+,t}\widetilde{u}(t,\xi)+|\xi|^2(t-a)^\gamma\widetilde{u}(t,\xi)=0,\\ (\mathfrak{D}^{\alpha-1}_{+,t}\widetilde{u})(0,\xi)=\widetilde{g}(\xi),\quad (\mathfrak{D}^{\alpha-2}_{+,t}\widetilde{u})(0,\xi)=\widetilde{h}(\xi),\quad t>a,\quad \xi\in\mathbb{R}^N.\end{cases}$$

According to [19, Example 4.16, p. 237], problem (4.1) has the following solution:

$$\tilde{u}(t,\xi)=\frac{\widetilde{g}(\xi)\left(\log\frac{t}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}\mathcal{E}_1(t,\xi)+\frac{\widetilde{h}(\xi)\left(\log\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\mathcal{E}_2(t,\xi).$$

Applying the inverse Fourier transformation, we get solution (4.2) of problem (4.1).

To prove the inclusion $\left(\log\frac{t}{a}\right)^{\alpha-2}u\in C(\mathbb{R}^+_a;\mathbb{H}^s(\mathbb{R}^N))$, note that the symbols $\mathcal{E}_1(t,\xi)$ and $\mathcal{E}_2(t,\xi)$ in (4.3) are uniformly bounded functions of the variable ξ (see Theorem 3.5). Then the operators $W^0_{\mathbb{R}^N,\mathcal{E}_k(t,\cdot)}$, $k=1,2$, are bounded in the space $\mathbb{H}^s(\mathbb{R}^N)=\mathbb{H}^s_2(\mathbb{R}^N)$ for all $s\in\mathbb{R}$.

About the uniqueness of a solution: If the Cauchy problem (4.1) has two solutions $u_1(t,x)$ and $u_2(t,x)$, their difference $u(t,x)=u_1(t,x)-u_2(t,x)$ solves the same problem (4.1), but with the vanishing Cauchy data $g(x)\equiv 0$ and $h(x)\equiv 0$. Then from the solution formula (4.2) it follows that $u(t,x)\equiv 0$. Therefore, $u_1(t,x)\equiv u_2(t,x)$, and the Cauchy problem (4.1) has a unique solution. □

4.2 Cauchy problem for $\mathfrak{D}^\alpha_{+,t}-\left(\log\frac{t}{a}\right)^\gamma\Delta_x$ on the domain $\mathbb{R}^+_a\times\Omega$

Let $\Omega\subseteq\mathbb{R}^N$ be a domain with the smooth boundary $\mathcal{S}=\partial\Omega$, Δ_x be the Laplace operator and let $\mathfrak{D}^\alpha_{+,t}$ be the modified Hadamard type fractional derivative of order $\alpha>0$, defined in (3.3a), (3.5c).

Theorem 4.2. *Let* $0<\alpha<1$, $\gamma>-\alpha$, $\lambda\in\mathbb{R}$, $1<p<\infty$, $s\in\mathbb{R}$. *Let* $\Omega\subset\mathbb{R}^N$ *be a compact domain with the smooth boundary* $\mathcal{S}:=\partial\Omega$. *Then the Cauchy problem for the time-fractional diffusion equation*

$$\begin{cases}\mathfrak{D}^\alpha_{+,t}u(t,x)-\left(\log\frac{t}{a}\right)^\gamma\Delta_x u(t,x)=0\,,\\ (\mathfrak{D}^{\alpha-1}_{+,t}u)(a,x)=g(x)\,,\quad t\in\mathbb{R}^+_a,\quad x\in\Omega,\\ u(t,\omega)\equiv 0\,,\quad t\in\mathbb{R}^+_a,\quad \omega\in\partial\Omega,\end{cases} \tag{4.4}$$

$$g\in\mathbb{L}_2(\Omega),\qquad \left(\log\frac{t}{a}\right)^{1-\alpha}u\in C(\mathbb{R}^+_a;\mathbb{L}_2(\mathbb{R}^N)),$$

for arbitrary $\varepsilon>0$, *has a unique solution of the form*

$$u(t,x)=\left(\log\frac{t}{a}\right)^{\alpha-1}\sum_{k=1}^\infty\frac{g_k}{\Gamma(\alpha)}E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-1}{\alpha}}\left(-\lambda_k\left(\log\frac{t}{a}\right)^{\gamma+\alpha}\right)e_k(x), \tag{4.5}$$

where $E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-1}{\alpha}}(z)$ *is the Kilbas–Saigo function (see Definition 3.4),* $\lambda_1,\lambda_2,\dots$ *are the eigenvalues of the Laplacian* Δ_x *on the domain* Ω, $e_1(x),e_2(x),\dots$ *are the corresponding eigenfunctions and*

$$g_k:=\int\limits_\Omega e_k(y)g(y)dy,\quad k\in\mathbb{N}:=1,2,\dots. \tag{4.6}$$

This solution has the following decay at infinity $(0 < m_1 < M_1 < \infty)$*:*

$$\|u(t,\cdot)\|_{\mathbb{L}_2(\Omega)} \leqslant \frac{\left(\log\frac{t}{a}\right)^{2(\alpha-1)}}{\Gamma^2(\alpha)\left(1+\lambda_1\left(\log\frac{t}{a}\right)^{\alpha+\gamma}\right)^2}\|g\|_{\mathbb{L}_2(\Omega)}. \tag{4.7}$$

Proof. Since $g \in \mathbb{L}_2(\Omega)$ and the eigenfunctions $\{e_k\}_{k\in N}$ of the Laplacian form an orthonormal basis in $\mathbb{L}_2(\Omega)$, the functions $u(t,x)$ and $g(x)$ have the expansions

$$u(t,x) = \sum_{k=1}^{\infty} u_k(t)e_k(x), \quad g(x) = \sum_{k=1}^{\infty} g_k e_k(x), \quad t \in \mathbb{R}_a^+, \quad x \in \Omega, \tag{4.8}$$

where g_k are defined in (4.6).

Substituting (4.8) into (4.4), we get the following system of Cauchy problems for the unknown functions $u_k(t)$:

$$\begin{cases} \mathcal{D}^{\alpha}_{+,t}u_k(t) + \lambda_k\left(\log\frac{t}{a}\right)^{\gamma}u_k(t) = 0, & t \in \mathbb{R}_a^+, \\ (\mathfrak{D}^{\alpha-1}_{+,t}u_k)(a) = g_k, & k = 1,2,\dots. \end{cases} \tag{4.9}$$

According to [19, Example 4.15, p. 237], problem (4.9) has the following solution:

$$u_k(t) = \frac{g_k}{\Gamma(\alpha)}\left(\log\frac{t}{a}\right)^{\alpha-1}E_{\alpha,1+\frac{\gamma}{\alpha},\frac{\gamma-1}{\alpha}}\left(-\lambda_k\left(\log\frac{t}{a}\right)^{\gamma+\alpha}\right). \tag{4.10}$$

Solution (4.5) of problem (4.4) is now obtained by inserting expressions for $u_k(t)$ from (4.10) into (4.8).

To prove the inclusions $u_\alpha(t,x) := \left(\log\frac{t}{a}\right)^{1-\alpha}u(t,x) \in C(\mathbb{R}_a^+;\mathbb{L}_2(\Omega))$, we note that the function $E_{\alpha,1+\frac{\gamma}{\alpha},\frac{\gamma-1}{\alpha}}\left(-\lambda_k\left(\log\frac{t}{a}\right)^{\gamma+\alpha}\right) \leqslant 1$ is uniformly bounded by 1 (see Theorem 3.5) and, applying the Parseval's identity, from (4.5), we derive

$$\begin{aligned}\sup_{t\geq a}\|u_\alpha(t,\cdot)\|^2_{\mathbb{L}_2(\Omega)} &= \sup_{t\geq a}\sum_{k=1}^{\infty}\frac{|g_k|^2}{\Gamma^2(\alpha)}\left|E_{\alpha,1+\frac{\gamma}{\alpha},\frac{\gamma-1}{\alpha}}\left(-\lambda_k\left(\log\frac{t}{a}\right)^{\gamma+\alpha}\right)\right|^2\|e_k(x)\|^2_{\mathbb{L}_2(\Omega)} \\ &\leqslant \sum_{k=1}^{\infty}\frac{|g_k|^2}{\Gamma^2(\alpha)} = \frac{\|g\|^2_{\mathbb{L}_2(\Omega)}}{\Gamma^2(\alpha)}.\end{aligned}$$

About the uniqueness of a solution: If the Cauchy problem (4.1) has two solutions $u_1(t,x)$ and $u_2(t,x)$, their difference $u(t,x) = u_1(t,x) - u_2(t,x)$ solves the same problem (4.4), but with the vanishing Cauchy condition $g(x) \equiv 0$. But then $g_1 = g_2 = \cdots = 0$ and from the solution formula (4.5) it follows that $u(t,x) \equiv 0$. Therefore, $u_1(t,x) \equiv u_2(t,x)$ and the Cauchy problem (4.4) has the unique solution.

To prove estimate (4.7), we proceed as follows: It is known that the eigenvalues λ_k of Dirichlet–Laplacian do not decrease, that is, $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots \nearrow +\infty$. Then, for the solution u, we have the following estimate:

$$\begin{aligned}\|u(t,\cdot)\|^2_{\mathbb{L}_2(\Omega)} &\leqslant \left(\log\frac{t}{a}\right)^{2(\alpha-1)}\sum_{k=1}^{\infty}\frac{|g_k|^2}{\Gamma^2(\alpha)}\left|E_{\alpha,1+\frac{\gamma}{\alpha},\frac{\gamma-1}{\alpha}}\left(-\lambda_k\left(\log\frac{t}{a}\right)^{\alpha+\gamma}\right)\right|^2\|e_k\|^2_{\mathbb{L}_2(\Omega)} \\ &\leqslant \left(\log\frac{t}{a}\right)^{2(\alpha-1)}\sum_{k=1}^{\infty}\frac{|g_k|^2}{\Gamma^2(\alpha)\left(1+\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\lambda_k\left(\log\frac{t}{a}\right)^{\alpha+\gamma}\right)^2} \\ &\leqslant \frac{\left(\log\frac{t}{a}\right)^{2(\alpha-1)}}{\Gamma^2(\alpha)\left(1+\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\lambda_1\left(\log\frac{t}{a}\right)^{\alpha+\gamma}\right)^2}\sum_{k=1}^{\infty}|\phi_k|^2 \\ &\leqslant \frac{\left(\log\frac{t}{a}\right)^{2(\alpha-1)}}{\Gamma^2(\alpha)\left(1+\lambda_1\left(\log\frac{t}{a}\right)^{\alpha+\gamma}\right)^2}\|g\|_{\mathbb{L}_2(\Omega)},\end{aligned}$$

and the proof is complete. □

4.3 Cauchy problem for $\mathfrak{D}^{\alpha}_{+,t} - \lambda(t-a)^{\gamma}\Delta_x$ on the set $(a,b] \times \mathbb{R}^N$

Let Δ_x be the Laplace operator on $\mathbb{R}^N$ and $\mathfrak{D}^{\alpha}_{+,t}$ be the modified Hadamard type fractional derivative of order $\alpha \in \mathbb{R}$ on the infinite interval $\mathbb{R}^+_a := (a, \infty)$ for some $a > 0$, defined in (3.3a) and (3.5c).

Theorem 4.3. *Let $m-1<\alpha<m$, $m = 1,2,\dots$, $\gamma > -\{\alpha\}$, $\lambda \in \mathbb{R}$, $1 < p < \infty$, $s \in \mathbb{R}$. Then the Cauchy problem for the time-fractional diffusion equation*

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}u(t,x) - \lambda(t-a)^{\gamma}\Delta_x u(t,x) = 0\,, \quad t \in (a,b], \quad x \in \mathbb{R}^N, \\ (\mathfrak{D}^{\alpha-k}_{+,t}u)(0,x) = g_k(x)\,, \quad g_k \in \mathbb{H}^s(\mathbb{R}^N), \quad k = 1,2,\dots,m, \end{cases} \tag{4.11}$$
$$(t-a)^{m-\alpha}u \in C((a,b];\mathbb{H}^s(\mathbb{R}^N)),$$

has a unique solution of the form

$$\begin{aligned} u(t,x) &= \sum_{k=1}^{m} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k-1)} \int\limits_{\mathbb{R}^N} e^{-ix\xi}\widetilde{g}_k(\xi)E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-k}{\alpha}}\left(-\lambda|\xi|^2(t-a)^{\gamma+\alpha}\right)d\xi \\ &= \sum_{k=1}^{m} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k-1)}(W^0_{\mathbb{R}^N,\mathcal{E}_k(t,\cdot)}g_k)(t,x), \end{aligned} \tag{4.12}$$

where $E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-k}{\alpha}}(z)$ is the Kilbas–Saigo function (see Definition 3.4) and

$$\mathcal{E}_k(t,\xi) := E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-k}{\alpha}}\left(-\lambda|\xi|^2(t-a)^{\gamma+\alpha}\right), \quad \widetilde{g}(\xi) = \int\limits_{\mathbb{R}^N} e^{is\xi}g(s)ds. \tag{4.13}$$

Proof. Applying the Fourier transformation in x to problem (4.11), we obtain the following equivalent Cauchy problem with the parameter:

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}\widetilde{u}(t,\xi) + |\xi|^2(t-a)^{\gamma}\widetilde{u}(t,\xi) = 0, \\ (\mathfrak{D}^{\alpha-k}_{+,t}\widetilde{u})(0,\xi) = \widetilde{g}_k(\xi), \quad k = 1,2,\dots,m, \quad t \in (a,b], \quad \xi \in \mathbb{R}^N. \end{cases} \tag{4.14}$$

According to [19, Example 4.4, p. 227], problem (4.14) has the following solution:

$$\tilde{u}(t,\xi) = \sum_{k=1}^{m} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k-1)}\widetilde{g}_k(\xi)E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-k}{\alpha}}\left(-\lambda|\xi|^2(t-a)^{\gamma+\alpha}\right).$$

Applying the inverse Fourier transformation, we obtain solution (4.12) of problem (4.11).

To prove the inclusion $(t-a)^{m-\alpha}u \in C((a,b];\mathbb{H}^s_p(\mathbb{R}^N))$, we note that the symbols $\mathcal{E}_1(t,\xi),\dots,\mathcal{E}_m(t,\xi)$ in (4.13) are uniformly bounded in ξ (see Theorem 3.5). Then the operators $W^0_{\mathbb{R}^N,\mathcal{E}_k(t,\cdot)}$, $k = 1,\dots,m$, are bounded in the space $\mathbb{H}^s(\mathbb{R}^N)$ for arbitrary $s \in \mathbb{R}$.

The uniqueness of the solution is proved as in Theorem 4.1. □

4.4 Cauchy problem for $\mathfrak{D}^{\alpha}_{+,t} - \lambda(t-a)^{\gamma}\Delta_x$ on the domain $(a,b] \times \Omega$

Let $\Omega \subseteq \mathbb{R}^N$ be a domain with the smooth boundary $\mathcal{S} = \partial\Omega$, Δ_x be the Laplace operator and let $\mathfrak{D}^{\alpha}_{+,t}$ be the modified Hadamard type fractional derivative of order $\alpha > 0$, defined in (3.3b), (3.5c).

Theorem 4.4. *Let $m-1<\alpha<m$, $\gamma > -\{\alpha\}$, $m = 1,2,\dots$, $\lambda \in \mathbb{R}$. Let $\Omega \subset \mathbb{R}^N$ be a compact domain with the smooth boundary $\mathcal{S} := \partial\Omega$. Then the Cauchy problem for the time-fractional diffusion equation*

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}u(t,x) - \lambda(t-a)^{\gamma}\Delta_x u(t,x) = 0\,, \quad t \in (a,b], \quad x \in \Omega, \\ (\mathfrak{D}^{\alpha-j}_{+,t}u)(0,x) = g_j(x), \quad j = 1,\dots,m, \end{cases} \tag{4.15}$$
$$g_j \in \mathbb{L}_2(\Omega), \quad j = 1,\dots,m, \quad (t-a)^{m-\alpha}u \in C((a,b];\mathbb{L}_2(\Omega)),$$

for arbitrary $\varepsilon > 0$, has a unique solution of the form

$$u(t,x) = \sum_{j=1}^{m}(t-a)^{\alpha-j}\sum_{k=1}^{\infty}\frac{g_{jk}}{\Gamma(\alpha-j-1)}E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-j}{\alpha}}\left(-\lambda_k(t-a)^{\gamma+\alpha}\right)e_k(x), \tag{4.16}$$

where $E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-j}{\alpha}}(z)$ is the Kilbas–Saigo function (see Definition 3.4), $\lambda_1,\lambda_2,\dots$ are the eigenvalues of the Laplacian Δ_x on the domain Ω, $e_1(x),e_2(x),\dots$ are the corresponding eigenfunctions and

$$g_{jk} := \int_{\Omega} e_k(y)g_j(y)dy, \quad j=1,2,\dots,m, \quad k\in\mathbb{N}. \tag{4.17}$$

This solution has the following decay at infinity:

$$\|u(t,\cdot)\|_{\mathbb{L}_2(\Omega)} \leqslant \frac{1}{(1+\lambda_1(t-a)^{\alpha+\gamma})^2}\sum_{j=1}^{m}\frac{(t-a)^{2(\alpha-j)}}{\Gamma^2(\alpha-j-1)}\|g_j\|^2_{\mathbb{L}_2(\Omega)}. \tag{4.18}$$

Proof. Since $g_j\in\mathbb{L}_2(\Omega)$ and the eigenfunctions $\{e_k\}_{k\in N}$ of the Laplacian form an orthonormal basis in $\mathbb{L}_2(\Omega)$, the functions $u(t,x)$ and $g_j(x)$ have the expansions

$$u(t,x)=\sum_{k=1}^{\infty}u_k(t)e_k(x), \quad g_j(x)=\sum_{k=1}^{\infty}g_{jk}e_k(x), \quad j=1,\dots,m, \quad t\in(a,b], \quad x\in\Omega,$$

where g_{jk} are defined in (4.17).

Substituting (4.19) into (4.15), we get the following infinite system of Cauchy problems for the unknown functions $u_k(t)$:

$$\begin{cases} \mathcal{D}^{\alpha}_{+,t}u_k(t)+\lambda_k(t-a)^{\gamma}u_k(t)=0, & t\in(a,b],\\ (\mathfrak{D}^{\alpha-j}_{+,t}u_k)(a)=g_{jk}, & j=1,\dots,m, \quad k=1,2,\dots. \end{cases} \tag{4.19}$$

According to [19, Example 4.4, p. 227], problem (4.19) has the following solution:

$$u_k(t)=\sum_{j=1}^{m}\frac{(t-a)^{\alpha-j}}{\Gamma(\alpha-j-1)}g_{jk}E_{\alpha,1+\frac{\gamma}{\alpha},1+\frac{\gamma-j}{\alpha}}\left(-\lambda_k(t-a)^{\gamma+\alpha}\right). \tag{4.20}$$

Solution (4.16) of problem (4.15) is now obtained by inserting expressions of $u_k(t)$ from (4.20) into (4.19).

To prove the inclusions $u_\alpha(t,x):=(t-a)^{m-\alpha}u(t,x)\in C((a,b];\mathbb{L}_2(\Omega))$, we note that the functions $E_{\alpha,1+\frac{\gamma}{\alpha},\frac{\gamma-j}{\alpha}}\left(-\lambda_k(t-a)^{\gamma+\alpha}\right)\leqslant 1$ are uniformly bounded by 1 (see Theorem 3.5) and, applying Parseval's identity, from (4.16) we derive:

$$\begin{aligned}\sup_{t\geq a}\|u_\alpha(t,\cdot)\|^2_{\mathbb{L}_2(\Omega)} &= \sup_{t\geq a}\sum_{j=1}^{m}\frac{(t-a)^{2(m-j)}}{\Gamma^2(\alpha-j-1)}\sum_{k=1}^{\infty}|g_{jk}|^2\left|E_{\alpha,1+\frac{\gamma}{\alpha},\frac{\gamma-1}{\alpha}}(-\lambda_k(t-a)^{\gamma+\alpha})\right|^2\|e_k(x)\|^2_{\mathbb{L}_2(\Omega)}\\ &\leqslant \sum_{j=1}^{m}\frac{(b-a)^{2(m-j)}}{\Gamma^2(\alpha-j-1)}\sum_{k=1}^{\infty}|g_{jk}|^2=\sum_{j=1}^{m}M_j\|g_j\|^2_{\mathbb{L}_2(\Omega)},\\ M_j &:= \frac{(b-a)^{2(m-j)}}{\Gamma^2(\alpha-j-1)}, \quad j=1,\dots,m.\end{aligned}$$

The uniqueness of the solution is proved as in Theorem 4.2.

To prove estimates (4.18), we proceed as follows: It is known that the eigenvalues λ_k of Dirichlet–Laplacian do not decrease, that is, $0<\lambda_1\leqslant\lambda_2\leqslant\cdots\leqslant\lambda_k\leqslant\cdots\nearrow+\infty$. Then, for the solution u, we have the following estimate:

$$\begin{aligned}\|u\,(t,\cdot)\,\|^2_{\mathbb{L}_2(\Omega)} &\leqslant \sum_{j=1}^{m}\frac{(t-a)^{2(\alpha-j)}}{\Gamma^2(\alpha-j-1)}\sum_{k=1}^{\infty}|g_{jk}|^2\left|E_{\alpha,1+\frac{\gamma}{\alpha},\frac{\gamma-j}{\alpha}}\left(-\lambda_k(t-a)^{\alpha+\gamma}\right)\right|^2\|e_k\|^2_{\mathbb{L}_2(\Omega)}\\
&\leqslant \sum_{j=1}^{m}\frac{(t-a)^{2(\alpha-j)}}{\Gamma^2(\alpha-j-1)}\sum_{k=1}^{\infty}\frac{|g_{jk}|^2}{\left(1+\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\lambda_k(t-a)^{\alpha+\gamma}\right)^2}\\
&\leqslant \frac{1}{\left(1+\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}\lambda_1(t-a)^{\alpha+\gamma}\right)^2}\sum_{j=1}^{m}\frac{(t-a)^{2(\alpha-j)}}{\Gamma^2(\alpha-j-1)}\sum_{k=1}^{\infty}|g_{jk}|^2\\
&\leqslant \frac{1}{(1+\lambda_1(t-a)^{\alpha+\gamma})^2}\sum_{j=1}^{m}\frac{(t-a)^{2(\alpha-j)}}{\Gamma^2(\alpha-j-1)}\|g_j\|^2_{\mathbb{L}_2(\Omega)},\end{aligned}$$

and the proof is complete. □

4.5 Cauchy problem for $\mathfrak{D}^{\alpha}_{+,t}-\lambda\Delta_x$ on the set $(a,b]\times\mathbb{R}^N$

Let Δ_x be the Laplace operator on $\mathbb{R}^N$ and let $\mathfrak{D}^{\alpha}_{+,t}$ be the modified Hadamard type fractional derivative of order $\alpha\in\mathbb{R}$ on the infinite interval $\mathbb{R}^+_a:=(a,\infty)$ for some $a>0$, defined in (3.3f) and (3.5d).

Theorem 4.5. *Let $m-1<\alpha<m$, $m=1,2,\dots$, $\lambda\in\mathbb{R}$, $1<p<\infty$, $s\in\mathbb{R}$. Then the Cauchy problem for the time-fractional diffusion equation*

$$\begin{cases}\mathfrak{D}^{\alpha}_{+,t}u(t,x)-\lambda\Delta_x u(t,x)=f(t,x)\,, & t\in(a,b],\quad x\in\mathbb{R}^N,\\ (\mathfrak{D}^{\alpha-k}_{+,t}u)(0,x)=g_k(x)\,, & g_k\in\mathbb{H}^s_p(\mathbb{R}^N),\quad 1\leqslant k\leqslant m,\end{cases}\tag{4.21}$$
$$f\in C((a,b];\mathbb{H}^s_p(\mathbb{R}^N)),\quad (t-a)^{m-\alpha}u\in C((a,b];\mathbb{H}^s_p(\mathbb{R}^N)),$$

has a unique solution of the form

$$\begin{aligned}\tilde{u}(t,\xi)&=\int_a^t(x-t)^{\alpha-1}\int_{\mathbb{R}^N}e^{-ix\xi}E_{\alpha,\alpha}\left[\lambda|\xi|^2(t-\tau)^{\alpha}\right]\widetilde{f}(\tau,\xi)d\xi d\tau\\
&\quad+\sum_{k=1}^{m}(t-a)^{\alpha-k}\int_{\mathbb{R}^N}e^{-ix\xi}E_{\alpha,\alpha-k+1}\left(-\lambda|\xi|^2(t-a)^{\alpha}\right)\widetilde{g}_k(\xi)d\xi\\
&=\int_a^t(x-t)^{\alpha-1}(W^0_{\mathbb{R}^N,\mathcal{E}_{\alpha,\alpha}(t,\cdot)}f)(\tau,x)d\tau+\sum_{k=1}^{m}(t-a)^{\alpha-k}(W^0_{\mathbb{R}^N,\mathcal{E}_{\alpha,\alpha-k+1}(t,\cdot)}g_k)(x),\end{aligned}\tag{4.22}$$

where $E_{\alpha,\alpha-k}(z)$ is the Mittag–Leffler function (see Definition 3.6) and

$$\mathcal{E}_{\alpha,\alpha-k}(t,\xi):=E_{\alpha,\alpha-k}\left(-\lambda|\xi|^2(t-a)^{\alpha}\right),\quad \widetilde{g}(\xi)=\int_{\mathbb{R}^N}e^{iy\xi}g(y)dy,\quad \widetilde{f}(t,\xi)=\int_{\mathbb{R}^N}e^{iy\xi}f(t,y)dy.\tag{4.23}$$

Proof. Applying the Fourier transformation in x to problem (4.21), we obtain the following equivalent Cauchy problem with the parameter:

$$\begin{cases}\mathfrak{D}^{\alpha}_{+,t}\widetilde{u}(t,\xi)+\lambda|\xi|^2\widetilde{u}(t,\xi)=\widetilde{f}(t,\xi),\\ (\mathfrak{D}^{\alpha-k}_{+,t}\widetilde{u})(0,\xi)=\widetilde{g}_k(\xi),\quad 1\leqslant k\leqslant m,\quad t\in(a,b],\quad \xi\in\mathbb{R}^N.\end{cases}\tag{4.24}$$

According to [19, Example 4.2, p. 225], problem (4.24) has the following solution:

$$\tilde{u}(t,\xi) = \int_a^t (x-t)^{\alpha-1} E_{\alpha,\alpha}\left[-\lambda|\xi|^2(t-\tau)^\alpha\right] \widetilde{f}(\tau,\xi)d\tau + \sum_{k=1}^m (t-a)^{\alpha-k} E_{\alpha,\alpha-k-1}\left(-\lambda|\xi|^2(t-a)^\alpha\right) \widetilde{g}_k(\xi).$$

By applying the inverse Fourier transformation, we get solution (4.22) of problem (4.21).

To prove the inclusion $(t-a)^{m-\alpha}u \in C((a,b];\mathbb{H}^s_p(\mathbb{R}^N))$, note that the symbols $\mathcal{E}_{\alpha,\alpha-k}(t,\xi)$, $0 \leqslant k \leqslant m-1$, in (4.23) and their derivatives in ξ, due to Lemma 3.7, satisfy conditions of Mikhlin–Hörmander–Lizorkin Theorem 2.3. Then the operators $W^0_{\mathbb{R}^N,\mathcal{E}_{\alpha,\alpha-k}(t,\cdot)}$, $0 \leqslant j \leqslant m-1$, are bounded in the space $\mathbb{H}^s_p(\mathbb{R}^N)$ for all $s \in \mathbb{R}$ and $1 < p < \infty$.

The uniqueness of the solution is proved as in Theorem 4.1. □

Corollary 4.6. *Let $m-1 < \alpha < m$, $m = 1,2,\dots$, $\lambda \in \mathbb{R}$, $1 < p < \infty$, $s \in \mathbb{R}$. If $f \in C^n((a+\varepsilon,b];\mathbb{H}^s_p(\mathbb{R}^N))$ for some $\varepsilon > 0$, $n = 1,2,\dots,\infty$, then the solution $u(x,t)$ of the Cauchy problem (4.21) has the property $u \in C^n((a+\varepsilon,b];\mathbb{H}^{s+2}_p(\mathbb{R}^N))$.*

Proof. We only need to explain that the symbols $\mathcal{E}_{\alpha,\alpha-k}(t,\xi)$, $0 \leqslant k \leqslant m-1$, in (4.23) and their derivatives $\xi^\omega\partial^\omega_\xi\mathcal{E}_{\alpha,\alpha-k}(t,\xi)$ have the asymptotics $\mathcal{O}(|\xi|^{-2})$ as $|\xi| \to \infty$ and $a+\varepsilon < t < b$. Therefore, the operators $W^0_{\mathbb{R}^N,\mathcal{E}_{\alpha,\alpha-k}(t,\cdot)}$ are bounded in the setting $\mathbb{H}^s_p(\mathbb{R}^N) \to \mathbb{H}^{s+2}_p(\mathbb{R}^N)$ (see Mikhlin–Hörmander–Lizorkin Theorem 2.3). □

4.6 Cauchy problem for $\mathfrak{D}^\alpha_{+,t} - \lambda\Delta_x$ on the set $(a,b] \times \Omega$

Let $\Omega \subseteq \mathbb{R}^N$ be a domain with the smooth boundary $\mathcal{S} = \partial\Omega$, Δ_x be the Laplace operator and let $\mathfrak{D}^\alpha_{+,t}$ be the modified Hadamard type fractional derivative of order $\alpha > 0$, defined in (3.3a), (3.5c).

Theorem 4.7. *Let $m-1 < \alpha < m$, $m = 1,2,\dots$, $\lambda \in \mathbb{R}$, $1 < p < \infty$, $s \in \mathbb{R}$ and let $\Omega \subset \mathbb{R}^N$ be a compact domain with the smooth boundary $\mathcal{S} := \partial\Omega$. Then the Cauchy problem for the time-fractional diffusion equation*

$$\begin{cases} \mathfrak{D}^\alpha_{+,t}u(t,x) - \lambda\Delta_x u(t,x) = f(t,x), \quad t \in (a,b], \quad x \in \Omega, \\ (\mathfrak{D}^{\alpha-k}_{+,t}u)(0,x) = g_k(x), \quad g_k \in \mathbb{H}^s_p(\Omega) \cap \mathbb{L}_2(\Omega) \quad 1 \leqslant k \leqslant m, \end{cases} \tag{4.25}$$
$$f \in C((a,b],\mathbb{L}_2(\Omega)), \quad (t-a)^{m-\alpha}u \in C((a,b];\mathbb{L}_2(\Omega)),$$

has a unique solution of the form

$$u(t,x) = \sum_{j=1}^\infty e_j(x)\int_a^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda\lambda_j(t-\tau)^\alpha\right]f_j(\tau)d\tau + \sum_{k=1}^m (t-a)^{\alpha-k}\sum_{j=1}^\infty E_{\alpha,\alpha-k}\left(-\lambda\lambda_j(t-a)^\alpha\right)g_{kj}e_j(x), \tag{4.26}$$

where $E_{\alpha,\alpha-k}(z)$ is the Mittag–Leffler function (see Definition 3.6), $\lambda_1,\lambda_2,\dots$ are the eigenvalues of the Laplacian Δ_x on the domain Ω, $e_1(x),e_2(x),\dots$ are the corresponding eigenfunctions and

$$g_{kj} := \int_\Omega e_j(y)g_k(y)dy, \quad f_j(t) := \int_\Omega e_j(y)f(t,y)dy, \quad 1 \leqslant k \leqslant m, \quad j \in \mathbb{N}. \tag{4.27}$$

Proof. Since $g_k \in \mathbb{L}_2(\Omega)$ and the eigenfunctions $\{e_j\}_{j\in\mathbb{N}}$ of the Laplacian form an orthonormal basis in $\mathbb{L}_2(\Omega)$, the functions $u(t,x)$ and $g_k(x)$ have the following expansions:

$$u(t,x)=\sum_{j=1}^{\infty}u_j(t)e_j(x),\quad f(t,x)=\sum_{j=1}^{\infty}f_j(t)e_j(x),\quad g_k(x)=\sum_{j=1}^{\infty}g_{kj}e_j(x),\tag{4.28}$$
$$1\leqslant k\leqslant m,\quad t\in(a,b],\quad x\in\Omega,$$

where g_{kj} and $f_j(t)$ are defined in (4.27).

Substituting (4.28) into (4.25), we get the following infinite system of Cauchy problems for the unknown functions $u_{kj}(t)$:

$$\begin{cases}\mathcal{D}^{\alpha}_{+,t}u_j(t)+\lambda\lambda_j u_j(t)=f_j(t), & t\in(a,b],\\ (\mathfrak{D}^{\alpha-k}_{+,t}u_j)(a)=g_{kj}, \quad 1\leqslant k\leqslant m, & j=1,2,\ldots.\end{cases}\tag{4.29}$$

According to [19, Example 4.2, p. 225], problem (4.29) has the following solution:

$$u_j(t)=\int_a^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left[\lambda\lambda_j(t-\tau)^{\alpha}\right]f_j(\tau)d\tau+\sum_{k=1}^{m}(t-a)^{\alpha-k}E_{\alpha,\alpha-k}\left(-\lambda\lambda_j(t-a)^{\alpha}\right)g_{kj}.\tag{4.30}$$

Solution (4.26) of problem (4.25) is now obtained by inserting expressions of $u_j(t)$ from (4.30) into (4.28).

To prove the inclusions $u_\alpha(t,x):=(t-a)^{m-\alpha}u(t,x)\in C((a,b];\mathbb{L}_2(\Omega))$, we note that the functions $E_{\alpha,\alpha-k}\left(-\lambda\lambda_k(t-a)^{\alpha}\right)\leqslant M_1$ are uniformly bounded (see Lemma 3.7) and, applying Parseval's identity, from (4.26) we derive

$$\begin{aligned}&\sup_{t\geqslant a}\|u_\alpha(t,\cdot)\|^2_{\mathbb{L}_2(\Omega)}\\ &\quad\leqslant M_1(b-a)^{2(m-\alpha)}\sup_{t\geqslant a}\int_\Omega\left|\int_a^t|t-\tau)^{\alpha-1}|f(\tau,x)|d\tau\right|^2dx+M_1(b-a)^2\sum_{k=1}^m\|g_k\|^2_{\mathbb{L}_2(\Omega)}\\ &\quad\leqslant M_1(b-a)^{2(m-\alpha)}\sup_{t\geqslant a}\left|\int_a^t|t-\tau|^{\alpha-1}d\tau\right|^2\int_\Omega\int_a^b|f(\tau,x)|^2d\tau dx+M_1(b-a)^2\sum_{k=1}^m\|g_k\|^2_{\mathbb{L}_2(\Omega)}\\ &\quad\leqslant M\left[\sup_{t\geqslant a}\|f(t,\cdot)\|^2_{\mathbb{L}_2(\Omega)}+\sum_{k=1}^m\|g_k\|^2_{\mathbb{L}_2(\Omega)}\right],\end{aligned}$$

since

$$\sup_{t\geqslant a}\left|\int_a^t|t-\tau)^{\alpha-1}|f(\tau,x)|d\tau\right|\leqslant\sup_{t\geqslant a}|f(t,x)|\int_a^b|t-\tau|^{\alpha-1}d\tau\leqslant M_2\sup_{t\geqslant a}|f(t,x)|.$$

The uniqueness of the solution is proved as in Theorem 4.2. □

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