Tunable Quantum Networks: Modeling design transparent and PT-symmetric quantum graphs

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OUTLINE
Part I: Linear waves in networks

- Quantum networks in physics
- Evolution equations on network
- Quantum graphs: Nonrelativistic case
- Quantum graphs: Relativistic case
- PT-symmetric quantum graphs
- Tunable wave propagation in quantum networks

Network science

Networks are used for modelling broad variety of complex systems:
From macromolecules to WWW, social, economic, political and ecological systems.

Networks can be modelled in terms of metric graphs.

Graph is characterized by its topology, a connection rule for graph bonds.

What is quantum Network?

No standard definition of quantum network.

Depends on the topic where network appears

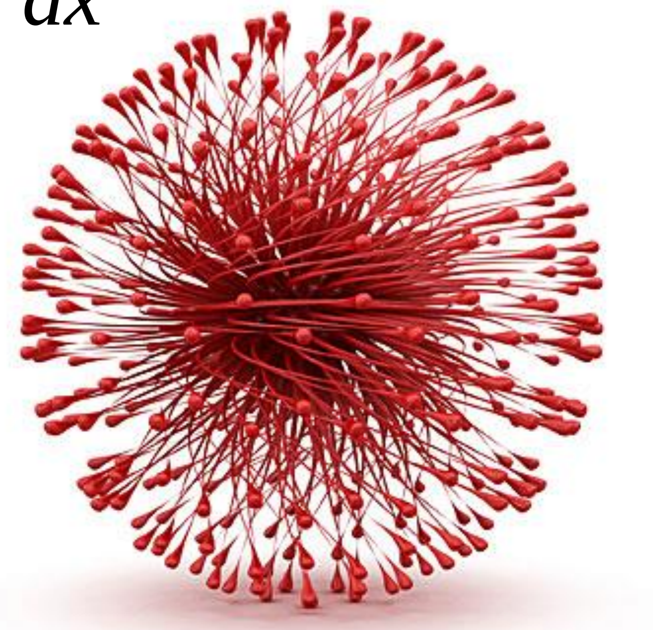
Our definition:

Any branched structure (network) where the particles/waves/phenomena are described in terms of quantum mechanical wave equations

Quantum Networks in Optics: Microwave Networks

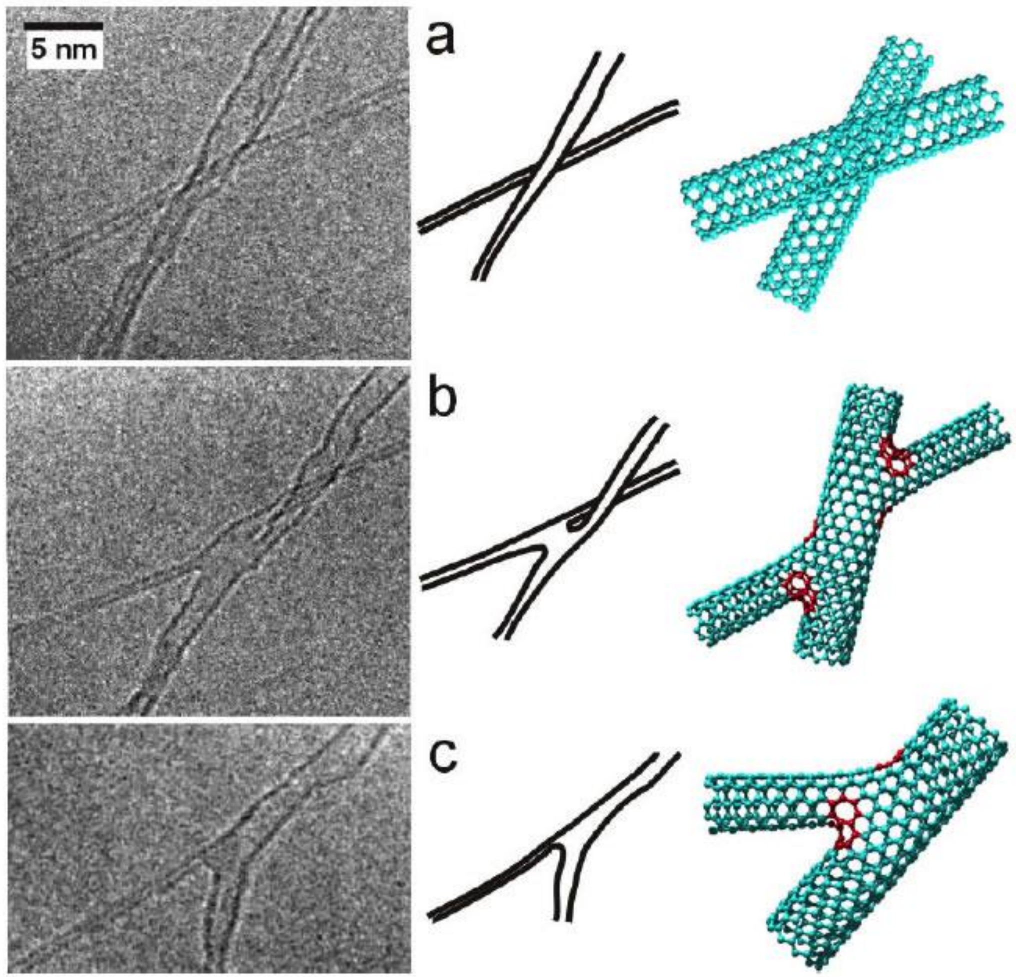
Wave transport in optical fibers is described by Helmholtz equation:

$$-\frac{d^2}{dx^2}\Psi = k^2\Psi$$



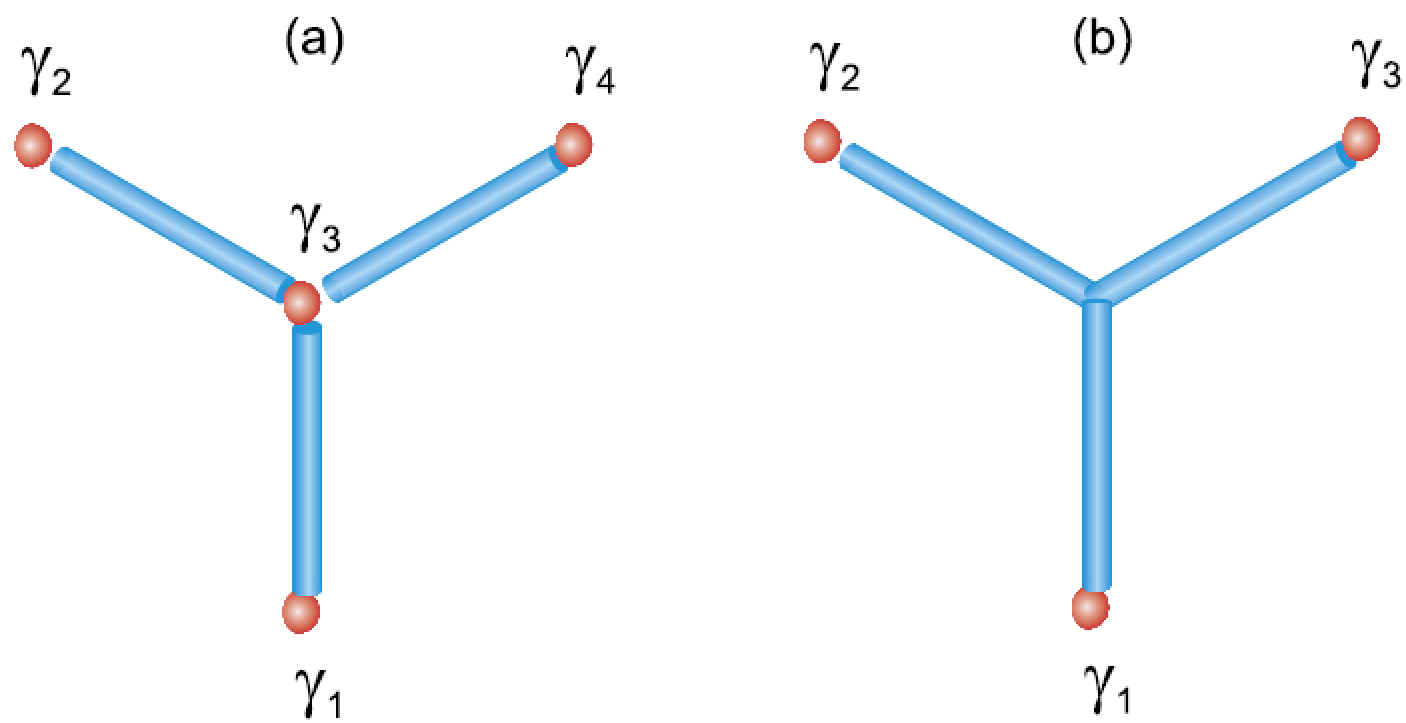
O . Hul *et al Phys. Rev.* E **69** 056205 (2004)

Quantum networks in condensed matter: Branched carbon nanotube

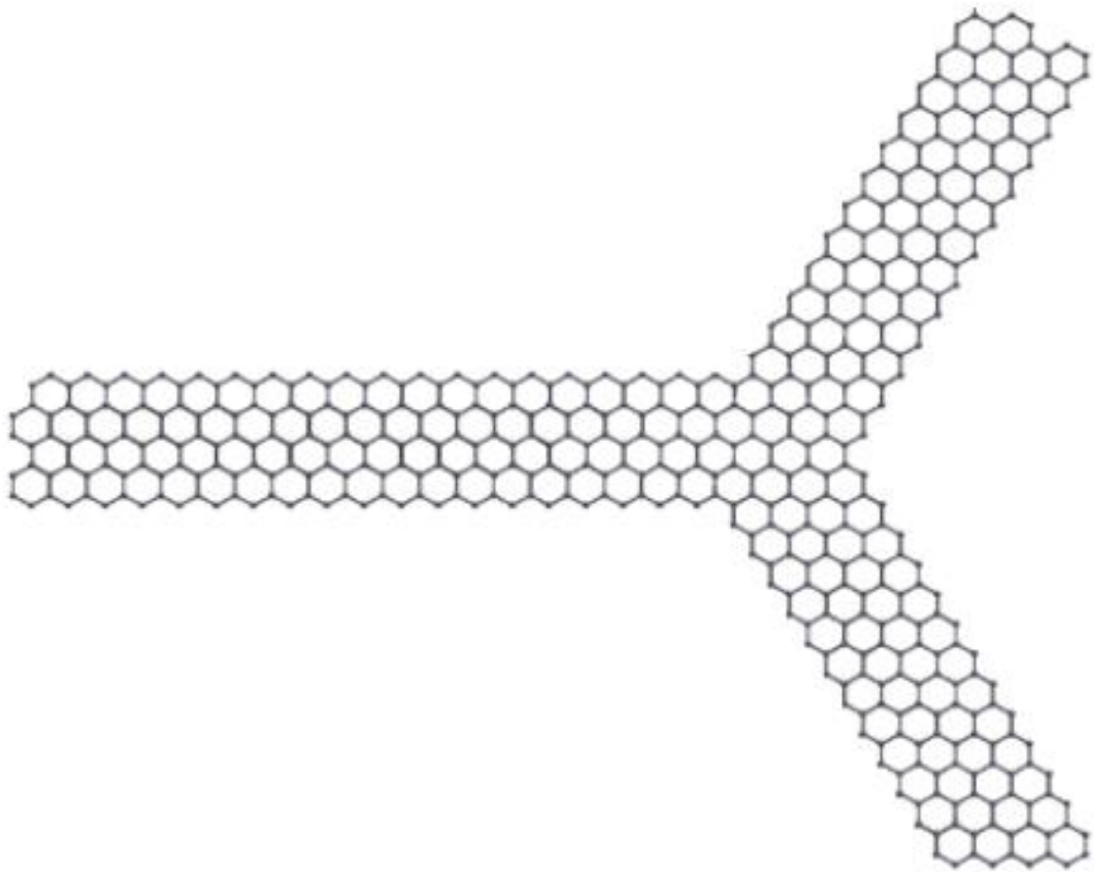


M. Terrones, F. Banhart, N. Grobert, J. C. Charlier, H. Terrones and P. M. Ajayan, Physical Review Letters 89, 75505, 2002.

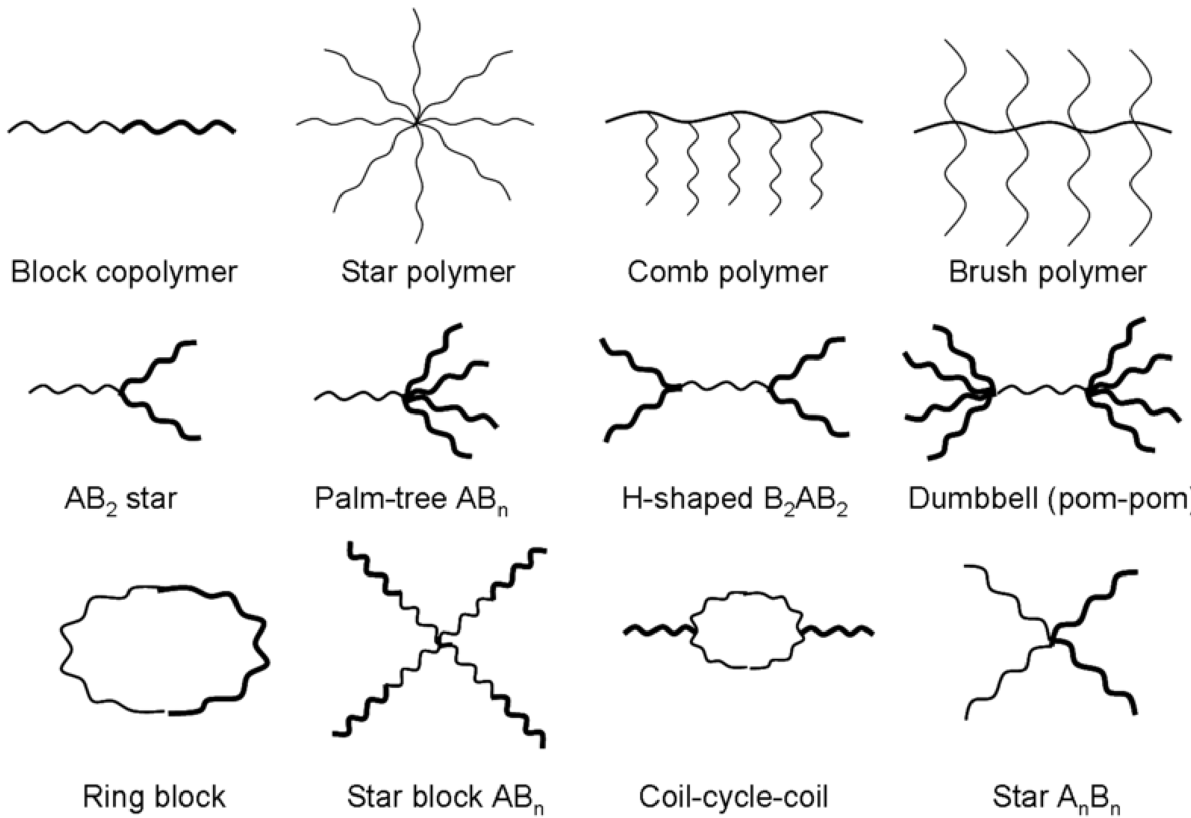
Quantum networks in condensed matter: Majorana wire networks



Quantum networks in condensed matter: Branched graphene nanoribbon



Quantum networks in polymers: Exciton dynamics in conducting polymers



Quantum networks in quantum information

IOP Publishing New J. Phys. 21 (2019) 033003 https://doi.org/10.1088/1367-2630/ab05f7

New Journal of Physics
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Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

PAPER

A quantum network stack and protocols for reliable entanglement-based networks

CrossMark

OPEN ACCESS

RECEIVED
9 December 2018

ACCEPTED FOR PUBLICATION
11 February 2019

PUBLISHED
15 March 2019

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Keywords: quantum networks, quantum communication, quantum internet

Abstract
We present a stack model for breaking down the complexity of entanglement-based quantum networks. More specifically, we focus on the structures and architectures of quantum networks and not on concrete physical implementations of network elements. We construct the quantum network stack in a hierarchical manner comprising several layers, similar to the classical network stack, and identify quantum networking devices operating on each of these layers. The layers responsibilities range from establishing point-to-point connectivity, over intra-network graph state generation, to inter-network routing of entanglement. In addition we propose several protocols operating on these layers. In particular, we extend the existing intra-network protocols for generating arbitrary graph states to ensure reliability inside a quantum network, where here reliability refers to the capability to compensate for devices failures. Furthermore, we propose a routing protocol for quantum routers which enables the generation of arbitrary graph states across network boundaries. This protocol, in correspondence with classical routing protocols, can compensate dynamically for failures of routers, or even complete networks, by simply re-routing the given entanglement over alternative paths. We also consider how to connect quantum routers in a hierarchical manner to reduce complexity, as well as reliability issues arising in connecting these quantum networking devices.

Quantum networks in quantum information

NATURE|Vol 453|19 June 2008|doi:10.1038/nature07127

The quantum internet

H. J. Kimble[1]

Quantum networks provide opportunities and challenges across a range of intellectual and technical frontiers, including quantum computation, communication and metrology. The realization of quantum networks composed of many nodes and channels requires new scientific capabilities for generating and characterizing quantum coherence and entanglement. Fundamental to this endeavour are quantum interconnects, which convert quantum states from one physical system to those of another in a reversible manner. Such quantum connectivity in networks can be achieved by the optical interactions of single photons and atoms, allowing the distribution of entanglement across the network and the teleportation of quantum states between nodes.

In the past two decades, a broad range of fundamental discoveries have been made in the field of quantum information science, from a quantum algorithm that places public-key cryptography at risk to a protocol for the teleportation of quantum states[1]. This union of quantum mechanics and information science has allowed great advances in the understanding of the quantum world and in the ability to control coherently individual quantum systems[2]. Unique ways in which quantum systems process and distribute information have been identified, and powerful new perspectives for understanding the complexity and subtleties of quantum dynamical phenomena have emerged.

In the broad context of quantum information science, quantum networks have an important role, both for the formal analysis and the physical implementation of quantum computing, communication and metrology[2–5]. A notional quantum network based on proposals in refs 4, 6 is shown in Fig. 1a. Quantum information is generated, processed and stored locally in quantum nodes. These nodes are linked by quantum channels, which transport quantum states from site to site with high fidelity and distribute entanglement across the entire network. As an extension of this idea, a 'quantum internet' can be envisaged; with only moderate processing capabilities, such an internet could accomplish tasks that are impossible in the realm of classical physics, including the distribution of 'quantum software'[7].

Apart from the advantages that might be gained from a particular algorithm, there is an important advantage in using quantum connectivity, as opposed to classical connectivity, between nodes. A network of quantum nodes that is linked by classical channels and comprises k nodes each with n quantum bits (qubits) has a state space of dimension $k2^n$, whereas a fully quantum network has an exponentially larger state space, 2^{kn}. Quantum connectivity also provides a potentially powerful means to overcome size-scaling and error-correlation problems that would limit the size of machines for quantum processing[8]. At any stage in the development of quantum technologies, there will be a largest size attainable for the state space of individual quantum processing units, and it will be possible to surpass this size by linking such units together into a fully quantum network.

A different perspective of a quantum network is to view the nodes as components of a physical system that interact by way of the quantum channels. In this case, the underlying physical processes used for quantum network protocols are adapted to simulate the evolution of quantum many-body systems[9]. For example, atoms that are localized at separate nodes can have effective spin–spin interactions catalysed by single-photon pulses that travel along the channels between the nodes[10]. This 'quantum wiring' of the network allows a wide range for the effective hamiltonian and for the topology of the resultant 'lattice'. Moreover, from this perspective, the extension of entanglement across quantum networks can be related to the classical problem of percolation[11].

These exciting opportunities provide the motivation to examine research related to the physical processes for translating the abstract illustration in Fig. 1a into reality. Such considerations are timely because scientific capabilities are now passing the threshold from a learning phase with individual systems and advancing into a domain of rudimentary functionality for quantum nodes connected by quantum channels.

In this review, I convey some basic principles for the physical implementation of quantum networks, with the aim of stimulating the involvement of a larger community in this endeavour, including in systems-level studies. I focus on current efforts to harness optical processes at the level of single photons and atoms for the transportation of quantum states reliably across complex quantum networks.

Two important research areas are strong coupling of single photons and atoms in the setting of cavity quantum electrodynamics (QED)[12] and quantum information processing with atomic ensembles[13], for which crucial elements are long-lived quantum memories provided by the atomic system and efficient, quantum interfaces between light and matter. Many other physical systems are also being investigated and are discussed elsewhere (ref. 2 and websites for the Quantum Computation Roadmap (http://qist.lanl.gov/qcomp_map.shtml), the SCALA Integrated Project (http://www.scala-ip.org/public) and Qubit Applications (http://www.qubitapplications.com)).

A quantum interface between light and matter

The main scientific challenge in the quest to distribute quantum states across a quantum network is to attain coherent control over the interactions of light and matter at the single-photon level. In contrast to atoms and electrons, which have relatively large long-range interactions for their spin and charge degrees of freedom, individual photons typically have interaction cross-sections that are orders of magnitude too small for non-trivial dynamics when coupled to single degrees of freedom for a material system.

The optical physics community began to address this issue in the 1990s, with the development of theoretical protocols for the coherent transfer of quantum states between atoms and photons in the setting of cavity QED[6,14,15]. Other important advances have been made in the past

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Metric graphs

A graph with the bonds which can be assigned length,

$$o < l_b : < D$$

is called metric graph

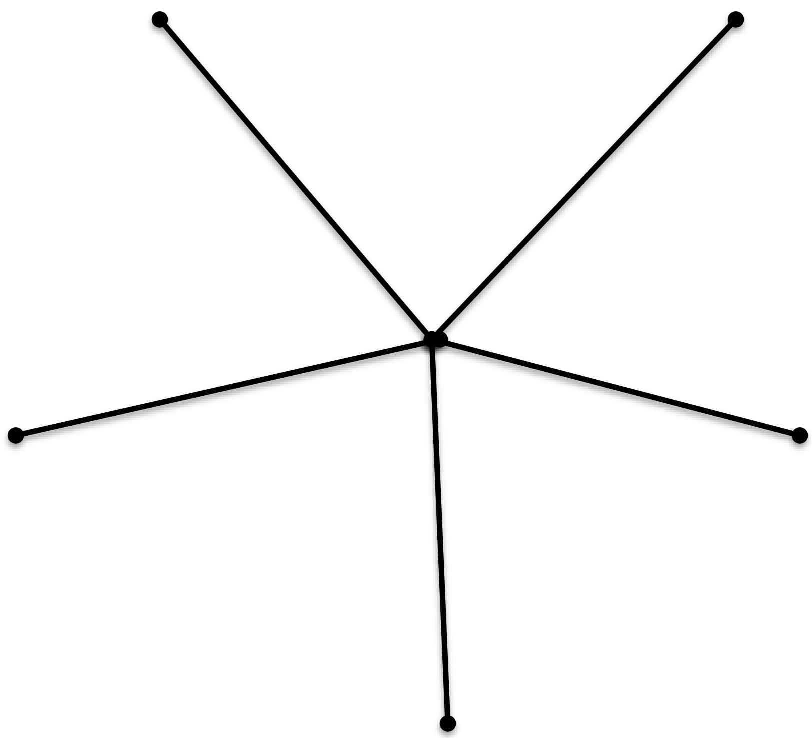
Graphs and their topology

The topology of the graph, that is, the way the vertices and bonds are connected is given in terms of the $V \times V$ connectivity matrix $C_{i,j}$ (sometimes referred to as the adjacency matrix) which is defined as:

$$C_{i,j} = C_{j,i} = \left\{ \begin{array}{l} 1 \text{ if } i,j \text{ are connected} \\ 0 \text{ otherwise} \end{array} \right\}, \; i,j = 1, ..., V.$$

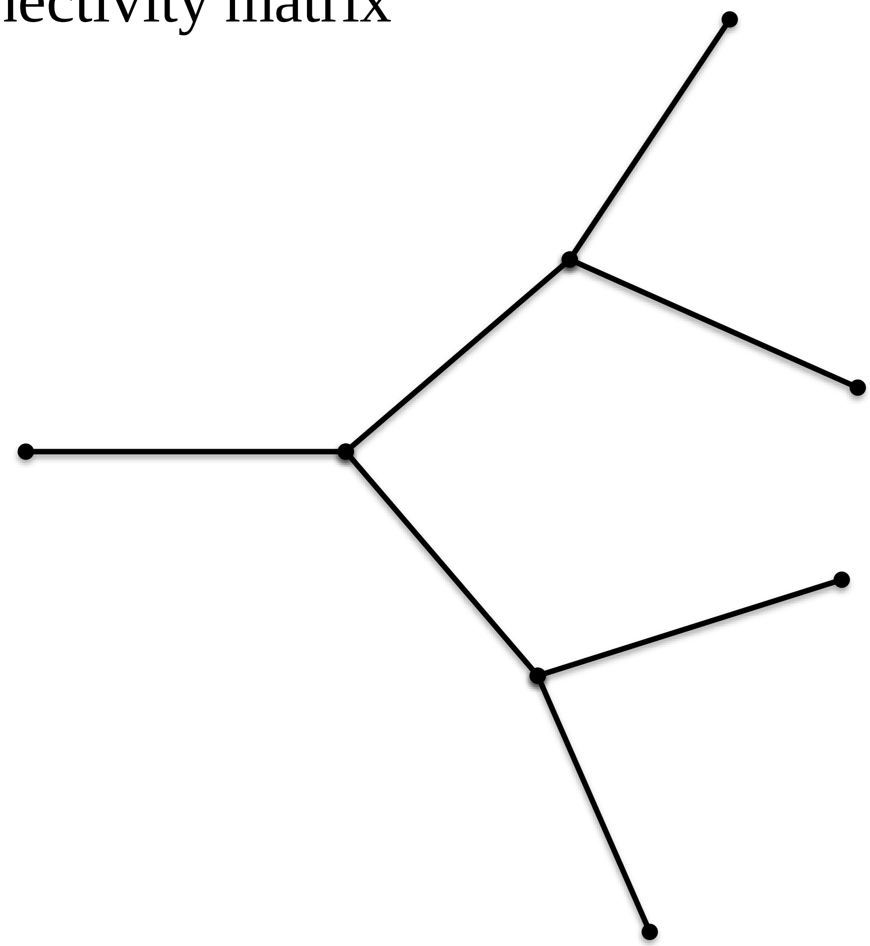
Constructing quantum graphs from finite interval (wires)

Metric graph as a collection of interval glues to each other according to connectivity matrix



Constructing quantum graphs from finite interval (wires)

Metric graph as a collection of interval glues to each other according to connectivity matrix



Evolution equation on graphs

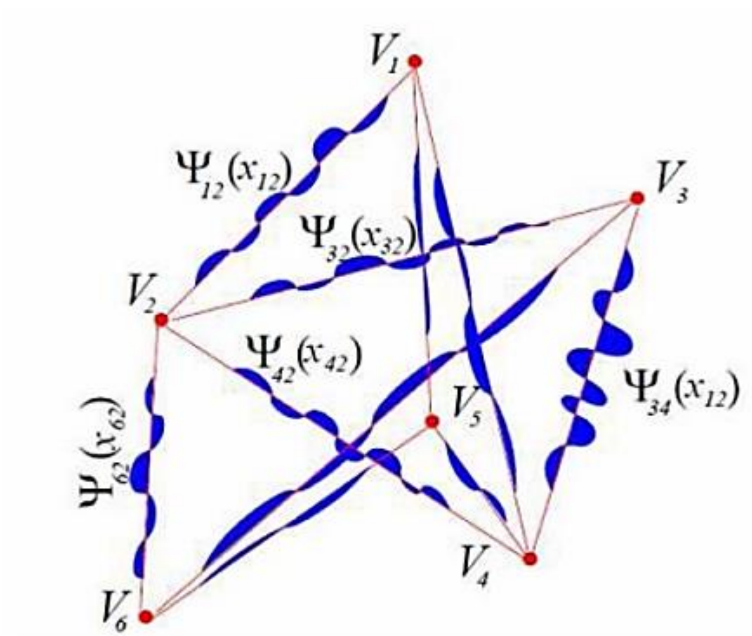
$$i\frac{\partial\psi}{\partial t} = H\psi$$

where H is the Shrödinger or Dirac operator

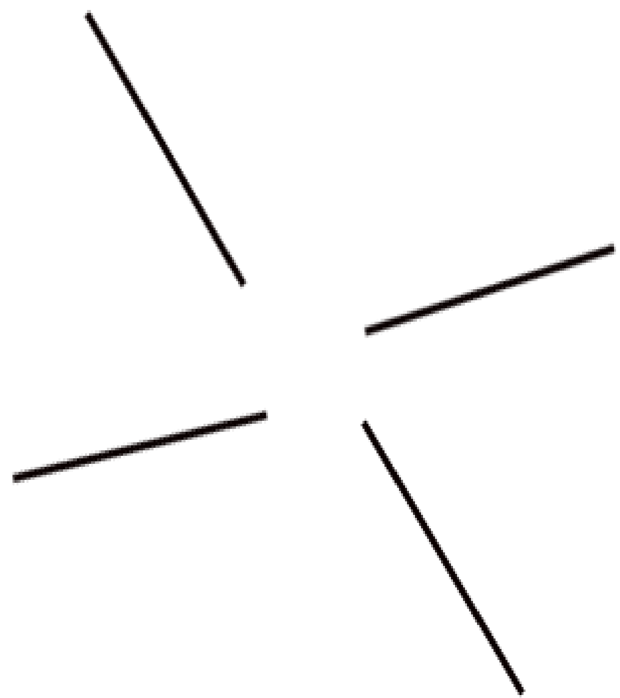
Wave equation on graphs: Wave function

Wave function Ψ is a B-component vector

$$\left(\Psi_{b_1}(x_{b_1}), \Psi_{b_2}(x_{b_2}), \dots, \Psi_{b_B}(x_{b_B})\right)^T$$



Wave equation on graphs: Vertex Boundary conditions



Differential operators on graphs

For given self-adjoint differential operator on graph D skew-Hermitian form can be constructed as

$$\Omega(\varphi,\phi)=\langle D\varphi,\phi\rangle-\langle \varphi,D\phi\rangle$$

V.Kostrykin, R.Schrader, J. Phys. A. **32** 595 (1999)

Boundary conditions

$$\mathbf{A}\psi(0) + \mathbf{B}\psi'(0) = 0$$

where A and B are two n×n matrices

V.Kostrykin, R.Schrader, J. Phys. A: Math. Gen. **32** (1999) 595–630.

The Schrödinger equation on graphs: Wave function

For each bond $b = (i, j)$ a coordinate $x_{i,j}$ which indicates the position along the bond is assigned. The variable $x_{i,j}$ takes the value 0 at the vertex i and the value $L_{i,j} \equiv L_{j,i}$ at the vertex j while $x_{j,i}$ is zero at j and $L_{i,j}$ at i. We have thus defined the length matrix $L_{i,j}$ with matrix elements different from zero, whenever $C_{i,j} \neq 0$ and $L_{i,j} = L_{j,i}$ for $b = 1, ..., B$.

The wavefunction Ψ is a B−component vector and can be written as

$$\left(\Psi_{b_1}(x_{b_1}), \Psi_{b_2}(x_{b_2}), ..., \Psi_{b_B}(x_{b_B})\right)^T$$

where the set $\{b_i\}_{i=1}^{B}$ consists of B different bonds

T. Kottos and U. Smilansky, Ann. Phys. 274, 76 (1999).

The Schrödinger equation on graphs: Boundary Conditions

The wave function must satisfy boundary conditions at the vertices, which ensure continuity (uniqueness) and current conservation. For every *i = 1, … , V* :

- $Continuity$:

$$\Psi_{i,j}(x)\Big|_{x=0} = \varphi_i, \qquad \Psi_{i,j}(x)\Big|_{x=L_{i,j}} = \varphi_j$$

For all $i < j$ and $C_{i,j} \neq 0$

- $Current\ conservation$

$$-\sum_{j<i} C_{i,j} \left.\frac{d\Psi_{i,j}(x)}{dx}\right|_{x=L_{i,j}} + \sum_{j>i} C_{i,j} \left.\frac{d\Psi_{i,j}(x)}{dx}\right|_{x=0} = \lambda_i \varphi_i$$

The parameters λ_i are free parameters which determine the type of the boundary conditions.

The special case of zero λ_i's, corresponds to Neumann boundary conditions. Dirichlet boundary conditions are introduced when all the $\lambda_i = \infty$.

T. Kottos and U. Smilansky, Ann. Phys. 274, 76 (1999).

The Schrödinger equation on graphs: Solutions

At any bond *b* = *(i, j)* the component *b* can be written in terms of its values on the vertices *i* and *j* as

$$\Psi_{i,j} = \frac{1}{\sin kL_{i,j}}\left(\varphi_i \sin\left[k\left(L_{i,j} - x\right)\right] + \varphi_j \sin kx\right)C_{i,j}, \qquad i < j.$$

The current conservation condition leads to

$$-\sum_{j<i} \frac{kC_{i,j}}{\sin\left(kL_{i,j}\right)}\left(-\varphi_j + \varphi_i \cos\left(kL_{i,j}\right)\right) + \sum_{j>i} \frac{kC_{i,j}}{\sin\left(kL_{i,j}\right)}\left(-\varphi_i \cos\left(kL_{i,j}\right) + \varphi_j\right) = \lambda_i \varphi_i, \qquad \forall i.$$

The Schrödinger equation on graphs: Eigenvalues

Spectral equation

$$\det\left(h_{i,j}(k)\right) = 0$$

where

$$h_{i,j} = \begin{cases} -\sum_{m \neq i} C_{i,m} \cot\left(kL_{i,m}\right) - \frac{\lambda_i}{k}, & i = j \\ C_{i,j}\left(\sin\left(kL_{i,j}\right)\right)^{-1}, & i \neq j. \end{cases}$$

Quantum star graph

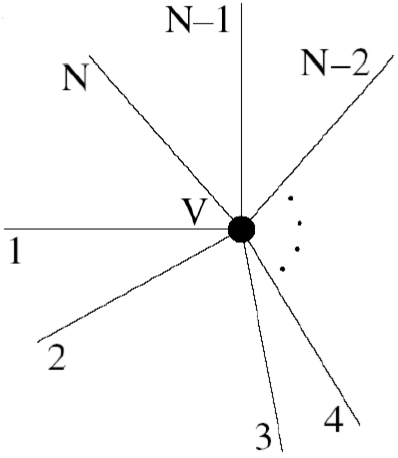
A graphs of the most simplest topology is so-called star-graph. It consist of three or more bonds connected at the single vertex which can be called central vertex. Other ones are called edge vertices. The eigenvalue problem for a star graph with N bonds is given by the following Schrödinger equation:

$$-i\frac{d^2}{dx^2}\phi_j(y) = k^2\phi_j(y), \qquad j = 1, \ldots, N.$$

We assign for each bond j a coordinate y_j which indicates the position along the bond and takes the value 0 at the vertex V and the value l_j at the edge vertex.

The boundary conditions for the star graph are

$$\begin{cases} \phi_1|_{y=0} = \phi_2|_{y=0} = \cdots = \phi_N|_{y=0}, \\ \phi_1|_{y=l_1} = \phi_2|_{y=l_2} = \cdots = \phi_N|_{y=l_N} = 0, \\ \sum\limits_{j=1}^{N} \frac{d}{dy}\phi_j\Big|_{y=0} = 0. \end{cases}$$



J.P.Keating, Contemp. Math., 415, 191 (2006)

Quantum star graph

The eigenvalues can be found by solving the following equation

$$\sum_{j=1}^{N} \cot(k l_j) = 0$$

where corresponding eigenfunctions are given as

$$\phi_j^{(n)}(y) = \frac{B_n}{\sin k_n l_j} \sin k_n (l_j - y)$$

with normalization coefficient

$$B_n = \sqrt{\frac{2}{\sum_j \frac{l_j - \sin 2k_n l_j}{\sin^2 k_n l_j}}}$$

Quantum transport

Probability current

$$J_k(x,t) = \frac{1}{2i}\left[\Psi_k^*(x,t)\,\frac{d\Psi_k}{dx} - \Psi_k(x,t)\,\frac{d\Psi_k^*}{dx}\right]$$

$$\Psi_k(x,t) = \sum_n e^{-iE_n t}\,\psi_k^{(n)}(x)$$

Quantum transport

Conductivity

$$\sigma_k(x) = \frac{1}{\omega}\int_0^\infty d\tau e^{-i\omega\tau}\langle[J_k(x,0), J_k(x,\tau)]\rangle$$

$$\langle[J_k(x,0), J_k(x,\tau)]\rangle$$

$$= \int_0^{L_k} dx[J_k(x,0)J_k(x,\tau) - J_k(x,0)J_k(x,\tau)]$$

PT-symmetric quantum mechanics

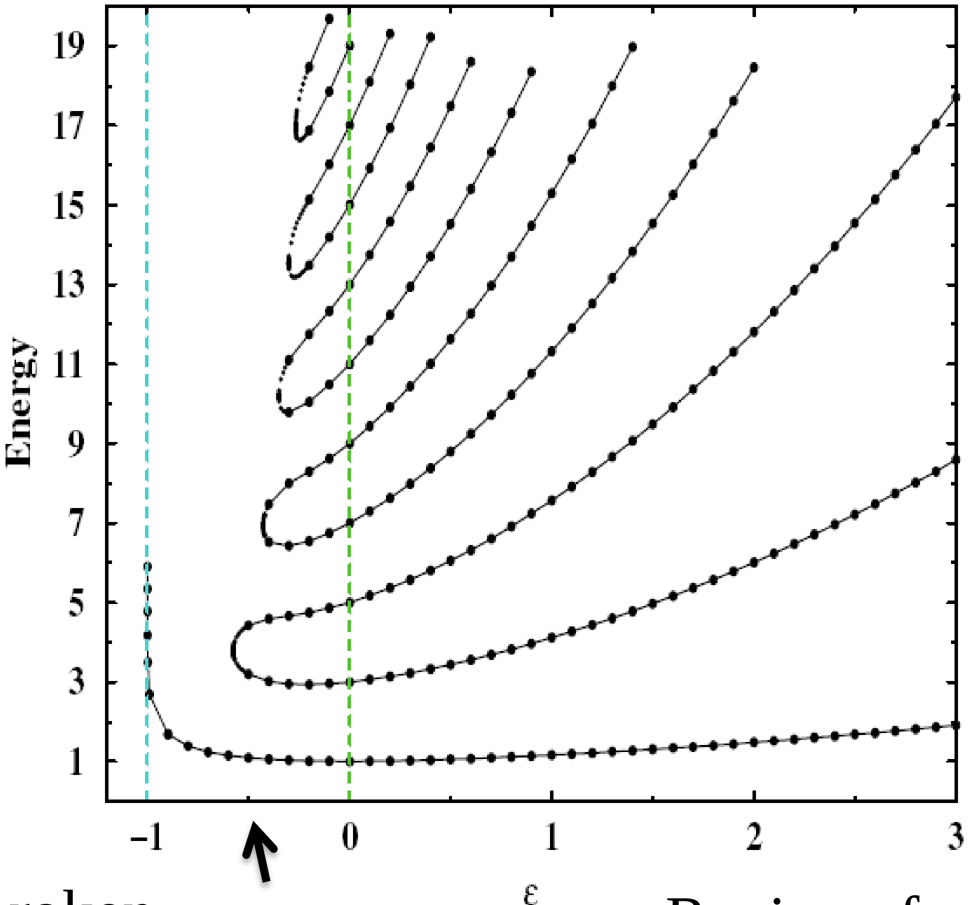
Since from the beginning of quantum physics people believed that to have real energy spectrum Hamiltonian operator should be Hermitian (self-adjoint). This fact was considered as necessary and enough condition for the realness of the spectrum. However such faith was broken in 1998 by Bender and Boettcher.

PT-symmetric quantum mechanics

In 1998, Bender and Boettcher [Phys. Rev. Lett. ***80*** *5243 (1998)] showed that quantum systems with a non-Hermitian Hamiltonian can have a set of eigenstates with real eigenvalues (a real spectrum).*

In other words, they found that the Hermiticity of the Hamiltonian is not a necessary condition for the realness of its eigenvalues, and new quantum mechanics can be constructed based on such Hamiltonians.

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$



Region of broken **PT** symmetry

PT phase transition

Region of unbroken **PT** symmetry

C. Bender and S. Boettcher, *PRL* **80**, 5243 (1998)

Examples of PT-symmetric systems

$$H = p^2 + x^{2K}(ix)^{\epsilon}$$

$$\mathcal{L} = \frac{1}{2}\,(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + g\phi^2(i\phi)^{\epsilon} \qquad (\epsilon \geq 0)$$

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2\,\frac{1}{2}i\bar{\psi}\partial\psi + \frac{1}{2}S'(\phi)\bar{\psi}\psi + \frac{1}{2}[S(\phi)]^2 =$$

$$= \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}i\bar{\psi}\partial\psi + \frac{1}{2}g(1+\epsilon)(i\phi)^{\epsilon}\bar{\psi}\psi - \frac{1}{2}g^2(i\phi)^{2+2\epsilon}$$

PT-Symmetric inner product

$$(f, g) = \int dx [PTf(x)] g(x)$$

$$\int dx g(x) [PTHf(x)] = \int dx Hg(x) [PTf(x)]$$

Introducing of PT-Symmetry in a quantum system

Similarly to that in Hermitian quantum mechanics, PT-symmetry in a quantum system can be introduced either via the **boundary conditions**, or **complex PT-symmetric potential.**

PT-symmetry in optics

Maxwell's equations reduce to the scalar Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{\omega}{c}\right)^2 \varepsilon(x,z)\right)E(x,z)=0$$

It formally coincides with the stationary Schrodinger equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\psi_k(x,z)-\frac{2m(V(x,z)-E_k)}{\hbar^2}\psi_k(x,z)=0$$

PT-symmetry in optics

Optical analog of the potential energy in quantum mechanics is the permittivity in optics: PT-symmetry condition for the optical system is defined as the condition imposed on the permittivity of the medium

$$\mathrm{Re}\,\varepsilon(\omega, x, z) = \mathrm{Re}\,(\omega, -x, -z)$$

$$\mathrm{Im}\,\varepsilon(\omega, x, z) = -\,\mathrm{Im}\,(\omega, -x, -z)$$

The stationary Schrödinger equation does not include the time dependence, and therefore the time reversal operation $\hat{T}$ is equivalent conjugation $\hat{K}$.

LETTERS
PUBLISHED ONLINE: 24 JANUARY 2010 | DOI: 10.1038/NPHYS1515

nature physics

Observation of parity–time symmetry in optics

Christian E. Rüter[1], Konstantinos G. Makris[2], Ramy El-Ganainy[2], Demetrios N. Christodoulides[2], Mordechai Segev[3] and Detlef Kip[1]*

One of the fundamental axioms of quantum mechanics is associated with the Hermiticity of physical observables[1]. In the case of the Hamiltonian operator, this requirement not only implies real eigenenergies but also guarantees probability conservation. Interestingly, a wide class of non-Hermitian Hamiltonians can still show entirely real spectra. Among these are Hamiltonians respecting parity–time (*PT*) symmetry[2–7]. Even though the Hermiticity of quantum observables was never in doubt, such concepts have motivated discussions on several fronts in physics, including quantum field theories[8], non-Hermitian Anderson models[9] and open quantum systems[10,11], to mention a few. Although the impact of *PT* symmetry in these fields is still debated, it has been recently realized that optics can provide a fertile ground where *PT*-related notions can be implemented and experimentally investigated[12–15]. In this letter we report the first observation of the behaviour of a *PT* optical coupled system that judiciously involves a complex index potential. We observe both spontaneous *PT* symmetry breaking and power oscillations violating left–right symmetry. Our results may pave the way towards a new class of *PT*-synthetic materials with intriguing and unexpected properties that rely on non-reciprocal light propagation and tailored transverse energy flow.

($\varepsilon > \varepsilon_{th}$), the spectrum ceases to be real and starts to involve imaginary eigenvalues. This signifies the onset of a spontaneous *PT* symmetry-breaking, that is, a 'phase transition' from the exact to broken-*PT* phase[7,20].

In optics, several physical processes are known to obey equations that are formally equivalent to that of Schrödinger in quantum mechanics. Spatial diffraction and temporal dispersion are perhaps the most prominent examples. In this work we focus our attention on the spatial domain, for example optical beam propagation in *PT*-symmetric complex potentials. In fact, such *PT* 'optical potentials' can be realized through a judicious inclusion of index guiding and gain/loss regions[7,12–14]. Given that the complex refractive-index distribution $n(x) = n_R(x) + in_I(x)$ plays the role of an optical potential, we can then design a *PT*-symmetric system by satisfying the conditions $n_R(x) = n_R(-x)$ and $n_I(x) = -n_I(-x)$.

In other words, the refractive-index profile must be an even function of position x whereas the gain/loss distribution should be odd. Under these conditions, the electric-field envelope E of the optical beam is governed by the paraxial equation of diffraction[13]:

$$i\frac{\partial E}{\partial z} + \frac{1}{2k}\frac{\partial^2 E}{\partial x^2} + k_0[n_R(x) + in_I(x)]E = 0$$

PT-symmetric quantum graph

Skew-Hermitian product on graph, which is defined for arbitrary differential operator, H as

$$\Omega(\psi,\phi) = \langle H\psi,\phi\rangle - \langle\psi,H\phi\rangle$$

$$\Omega(\psi,\phi) = -\sum_{j}^{N}\left[\phi^{*}{}_{j}(0)\frac{d\psi_j(0)}{dx} - \psi_j(0)\frac{d\phi^{*}{}_{j}(0)}{dx}\right] +$$

$$+\sum_{j}^{N}\left[\phi^{*}{}_{j}(L)\frac{d\psi_j(L)}{dx} - \psi_j(L)\frac{d\phi^{*}{}_{j}(L)}{dx}\right] = 0$$

Boundary conditions I

$$\psi_1(0) = \psi_2(0) = \psi_3(0),$$
$$\left.\frac{\partial \psi_1}{\partial x}\right|_{x=L_1} + \left.\frac{\partial \psi_2}{\partial x}\right|_{x=L_2} + \left.\frac{\partial \psi_3}{\partial x}\right|_{x=L_3} = 0,$$
$$\psi_j(L_j) = 0, \; j = 1, 2, 3.$$

Boundary conditions II

$$\left.\frac{\partial \psi_1}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_2}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_3}{\partial x}\right|_{x=0},$$
$$\psi_1(L_1) + \psi_2(L_2) + \psi_3(L_3) = 0,$$
$$\left.\frac{\partial \psi_j}{\partial x}\right|_{x=L_j} = 0, \;\; j = 1, 2, 3.$$

D. U. Matrasulov, K. K. Sabirov, J. R. Yusupov, J. Phys. A **52** 155302 (2019)

PT-symmetric quantum graph

Secular equation for finding energy spectrum

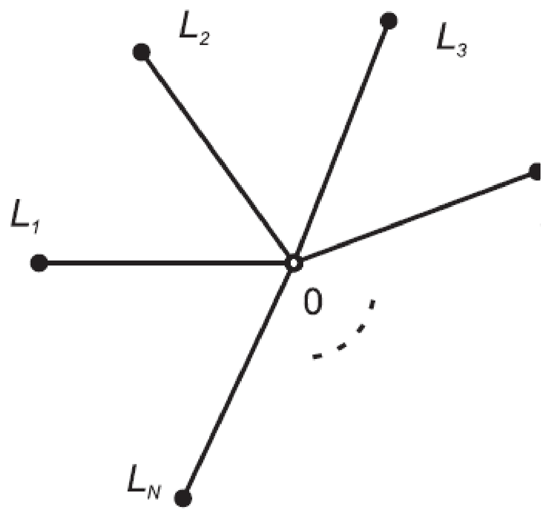
$$e^{ikL_1}\left(1-e^{2ikL_2}\right)\left(1-e^{2ikL_3}\right)+e^{ikL_2}\left(1-e^{2ikL_1}\right)\left(1-e^{2ikL_3}\right)+$$

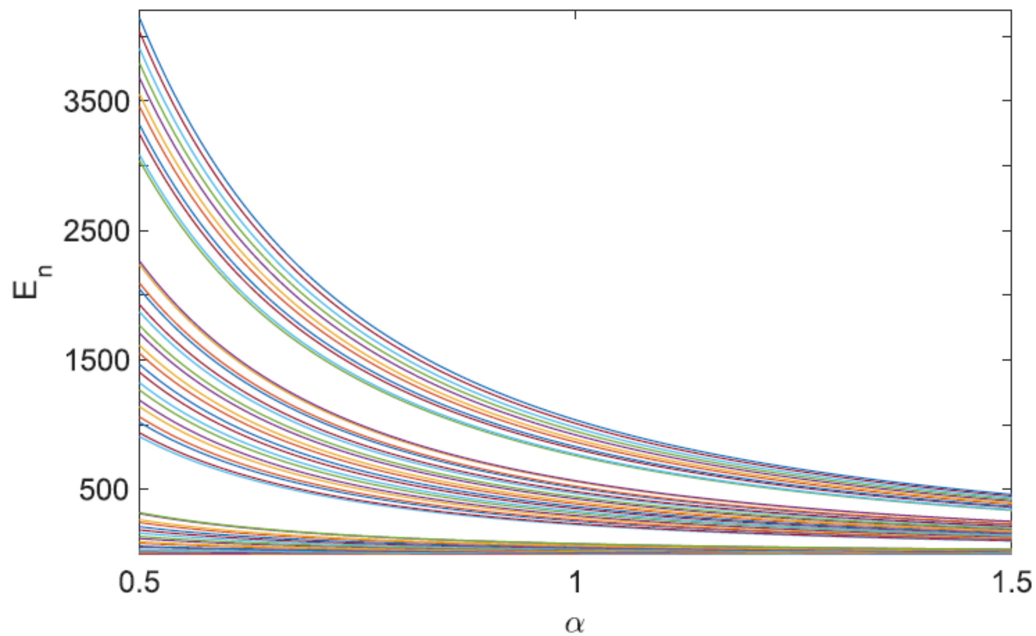
$$+e^{ikL_3}\left(1-e^{2ikL_1}\right)\left(1-e^{2ikL_2}\right)=0$$

$$\psi_j(x,k_n)=B\frac{\sin k_n\left(L_j-x\right)}{\sin k_n L_j}$$

D. U. Matrasulov, K. K. Sabirov, J. R. Yusupov, J. Phys. A **52** 155302 (2019)

PT-symmetric quantum graphs





D. U. Matrasulov, K. K. Sabirov, J. R. Yusupov, J. Phys. A **52** 155302 (2019)

Breaking of Kirchhoff rule

Total current at the vertex (x = 0)

$$J(0,t) = J_1(0,t) + J_2(0,t) + J_3(0,t)$$

$$J_j(0,t) = \frac{i}{2}\left[\psi_j(0,t)\partial_x\psi^*{}_j(0,t) - \partial_x\psi_j(0,t)\psi^*{}_j(0,t)\right]$$

$$\psi_j(x,t) = \sum_n C_n e^{-ik^2{}_n t}\phi_j(x,k_n)$$

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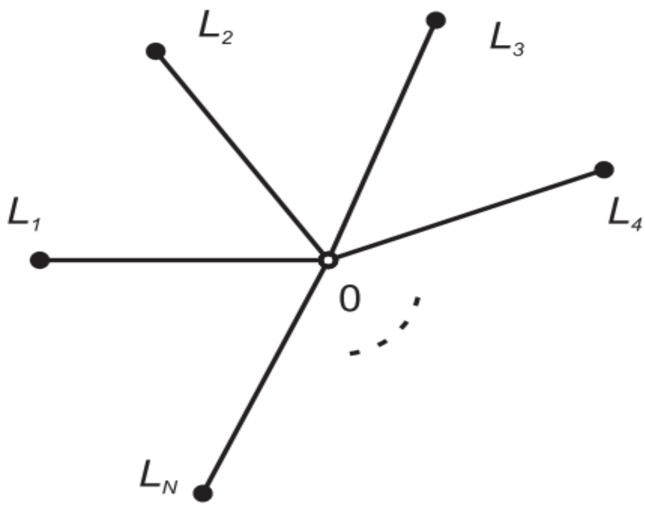
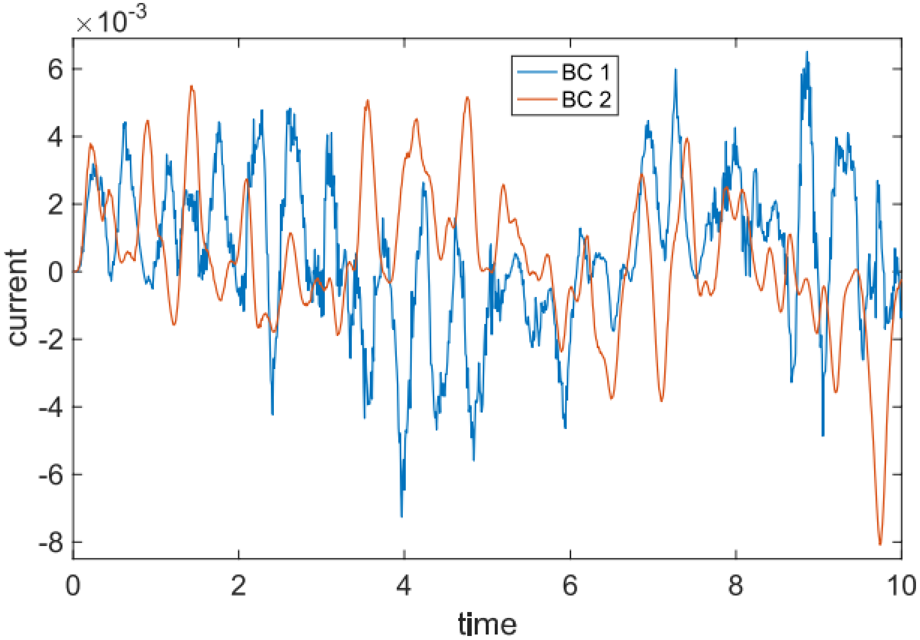
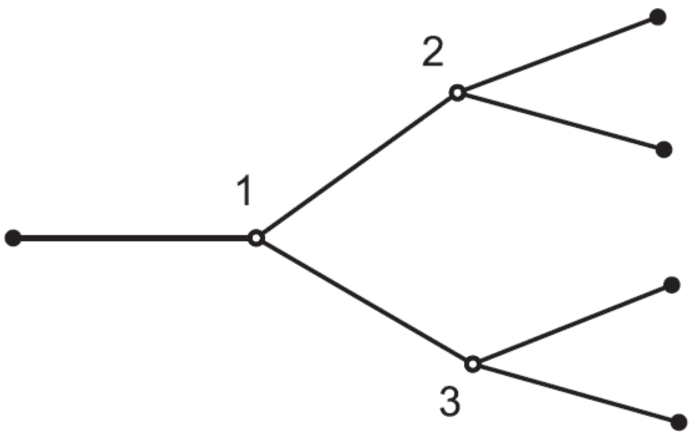
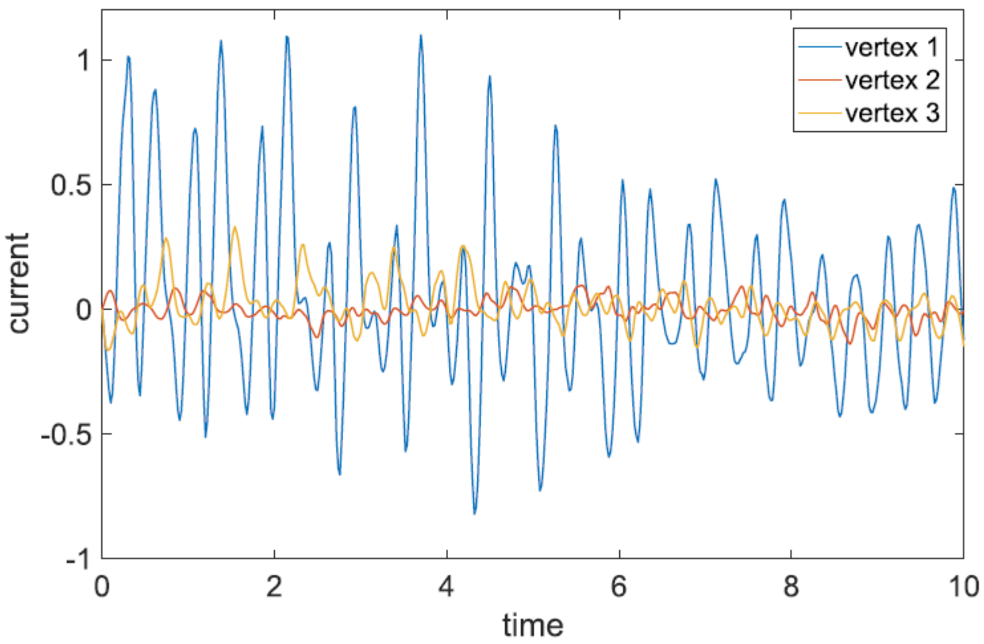
Breaking of Kirchhoff's rule





PT-symmetric quantum graphs





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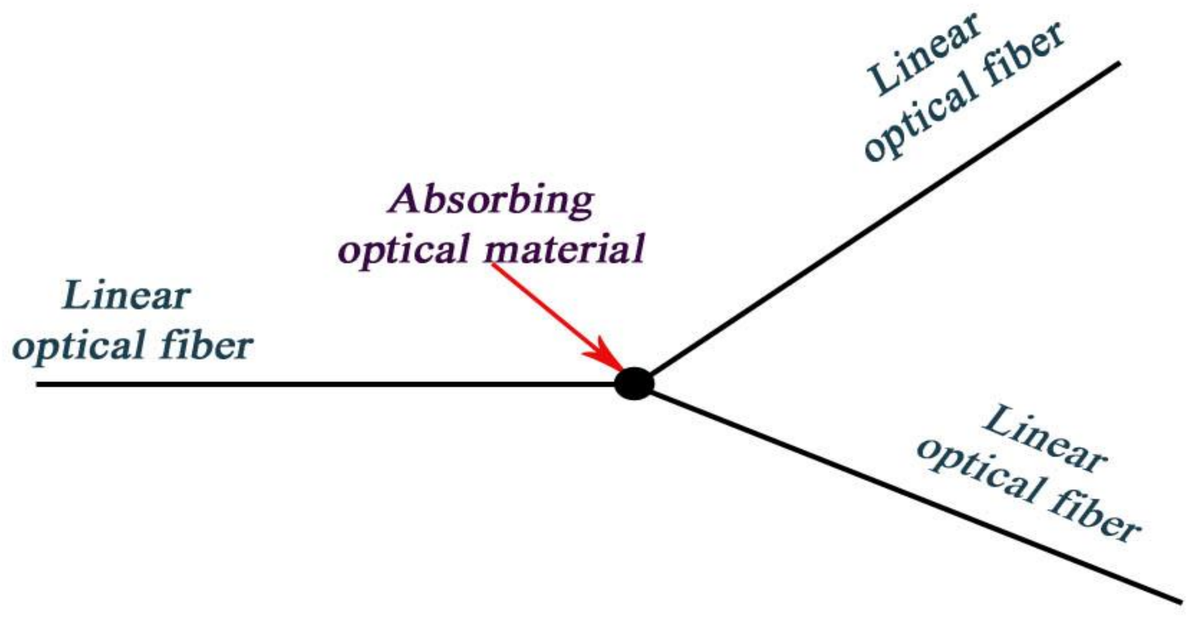
Experimental realization



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Reflectionless wave motion on graphs

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Transparent boundary conditions

- For a given finite domain, Ω, the TBC sare imposed in such a way that the solution of a PDE in Ω corresponds to that in the whole space, i.e., the wave/particle moving inside/outside the domain does not ‘see’ the boundary of the domain.

- Then such boundary conditions provide absence of the back scattering at the given point (or domain boundary) makes it transparent.

Transparent quantum graphs: Reflectionless wave propagation in quantum networks

Absence of backscattering at the graph vertices makes the graph transparent. Mathematically, such transparency can be provided by imposing so-called reflectionless boundary conditions at the graph vertex.

Transparent boundary conditions

- **<u>The general procedure for constructing transparent boundary conditions on a real line:</u>**
- 1. Split the original PDE evolution problem into coupled equations: the interior and exterior problems.
- 2. Apply a Laplace transformation to exterior problems on Ωext.
- 3. Solve (explicitly, numerically) the ordinary differential equations in the spatial unknownx.
- 4. Allow only “outgoing” waves by selecting the decaying solution as$x \to \pm\infty$.
- 5. Match Dirichlet and Neumann values at the artificial boundary.
- 6. Apply (explicitly, numerically) the inverse Laplace transformation

Transparent quantum networks:

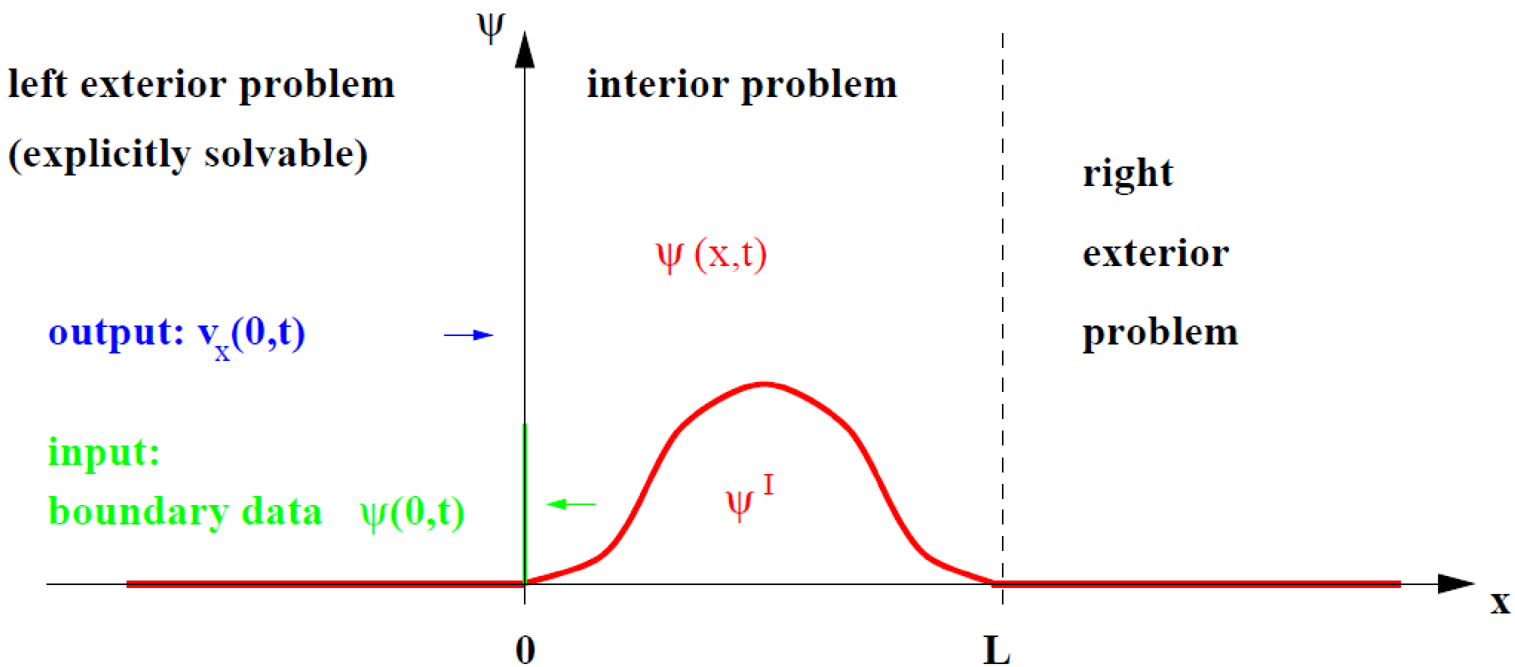


FIGURE 1. Schrödinger equation: Construction idea for transparent boundary conditions

M. Ehrhardt and A. Arnold, Discrete Transparent Boundary Conditions for the Schrödinger Equation, *Rivista di Mathematica della Universita di Parma*, Volume 6, Number 4 (2001), 57-108.

Transparent boundary condition

Interior problem:

$$i\partial_t \Psi = -\frac{1}{2}\partial_x^2\, \Psi + V(x,t)\Psi, \quad 0 < x < L, t > 0$$
$$\Psi(x,0) = \Psi^I(x)$$
$$\partial_x \Psi(0,t) = (T_0\Psi)(0,t)$$
$$\partial_x \Psi(L,t) = (T_L\Psi)(L,t)$$

$T_{0,L}$ denote the Dirichlet-to-Neumann maps at the boundaries.

Transparent boundary conditions

$T_{0,L}$ are obtained by solving the two exterior problems:

$$
\begin{aligned}
& i\partial_t v = -\tfrac{1}{2}\partial_x^2 v + V_L v, \quad x > L, \quad t > 0 \\
& v(x,0) = 0 \\
& v(L,t) = \Phi(t),\ t > 0, \\
& \Phi(0) = 0 \\
& v(\infty,t) = 0, \\
& (T_L\Phi)(t) = \partial_x v(L,t),
\end{aligned}
$$

and analogously for T_0.

Transparent boundary conditions

An inverse Laplace transformation yields the right TBC at $\$x = L\$$:

$$\partial_x \Psi(x = L, t) = -\sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} e^{-iV_L t} \frac{d}{dt} \int_0^t \frac{\Psi(L,\tau) e^{iV_L \tau}}{\sqrt{t-\tau}} d\tau$$

Similarly, the left TBC at $x = 0$ is obtained as

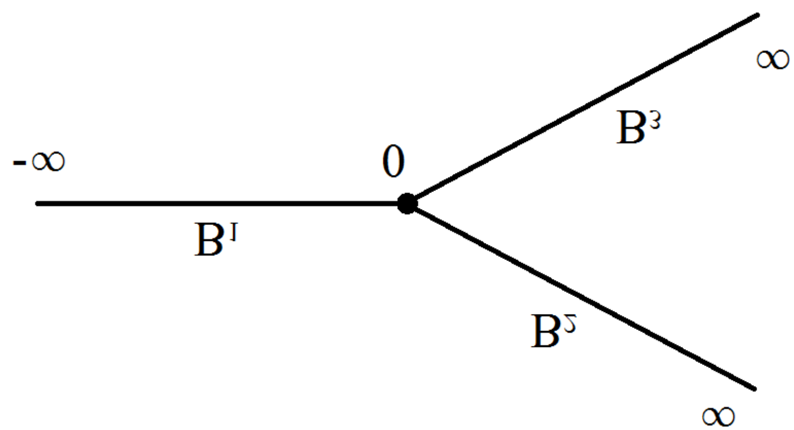
$$\partial_x \Psi(x = 0, t) = -\sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{4}} \frac{d}{dt} \int_0^t \frac{\Psi(L,\tau)}{\sqrt{t-\tau}} d\tau.$$

Transparent quantum networks:

Time-dependent Schrödinger equation for star graph with 3 bonds (in units $\hbar = m = 1$)

$$i\partial_t \Psi_b = -\frac{1}{2}\partial_x^2 \Psi_b, \qquad b = 1,2,3$$

The coordinates assigned to bond B_1 is $x \in (-\infty, 0)$ and B_$(1,2)$ are $x \in (0, \infty)$.



Transparent quantum networks:

Interior problem for B_1:

$$i\partial_t \Psi_1 = -\frac{1}{2}\partial_x^2 \Psi_1, \quad x < 0,\ t > 0$$

$$\Psi_1(x,0) = \Psi^I(x)$$

$$\partial_x \Psi_1(0,t) = (T_+\Psi_1)(0,t)$$

Transparent quantum networks

Exterior problems for $B_{2,3}$:

$$i\partial_t \Psi_{2,3} = -\frac{1}{2}\partial_x^2 \Psi_{2,3}, \quad x > 0, \quad t > 0$$

$$\Psi_{2,3}(x,0) = 0$$

$$\Psi_{2,3}(0,t) = \Phi_{2,3}(t), \quad t > 0, \quad \Phi_{2,3}(0) = 0$$

$$(T_+ \Phi_{2,3})(t) = \partial_x \Psi_{2,3}(0,t)$$

J. R. Yusupov, K. K. Sabirov, M. Ehrhardt, and D. U. Matrasulov, Phys. Lett. A **383**, 2382 (2019).

Transparent quantum networks

The Laplace transformed current conservation (at x = 0) takes the form

$$\frac{\partial}{\partial x}\hat{\Psi}_1 = \frac{\alpha_1}{\alpha_2}\frac{\partial}{\partial x}\hat{\Psi}_2 + \frac{\alpha_1}{\alpha_3}\frac{\partial}{\partial x}\hat{\Psi}_3$$
$$= -\sqrt[+]{-2is}\,\alpha_1^2\left(\frac{1}{\alpha_2^2}+\frac{1}{\alpha_3^2}\right)$$

Using the inverse transform we have

$$\frac{\partial}{\partial x}\Psi_1(x=0,t) = A_1\sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{4}}\frac{d}{dt}\int_0^t \frac{\Psi_1(0,\tau)}{\sqrt{t-\tau}}\tau$$

where $A_1 = \alpha_1^2(\alpha_2^{-2}+\alpha_3^{-2})$.

J. R. Yusupov, K. K. Sabirov, M. Ehrhardt, and D. U. Matrasulov, Phys. Lett. A **383**, 2382 (2019).

Transparent quantum networks

Continuity condition:

$$\alpha_1 \Psi_1(0,t) = \alpha_2 \Psi_2(0,t) = \alpha_3 \Psi_3(0,t)$$

Current conservation condition:

$$\frac{1}{\alpha_1}\partial_x \Psi_1(x=0,t) = \frac{1}{\alpha_2}\partial_x \Psi_2(x=0,t) + \frac{1}{\alpha_3}\partial_x \Psi_3(x=0,t)$$

Condition for transparency the continuity and current conservation:

$$\frac{1}{\alpha_1^2} = \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}.$$

J. R. Yusupov, K. K. Sabirov, M. Ehrhardt, and D. U. Matrasulov, Phys. Lett. A **383**, 2382 (2019).

RELATIVISTIC QUASIPARTICLES IN TRANSPARENT QUANTUM GRAPHS

Dirac equation ($\hbar = c = 1$):

$$i\partial_t \phi_j = -i\,\partial_x \chi_j + m\phi_j,$$
$$i\partial_t \chi_j = -i\,\partial_x \phi_j - m\chi_j.$$

Vertex boundary conditions: Weight continuity

$$\alpha_1\phi_1(0,t) = \alpha_2\phi_2(0,t) = \alpha_3\phi_3(0,t),$$

Vertex boundary conditions: Kirchoff rules

$$\frac{1}{\alpha_1}\chi_1(0,t) = \frac{1}{\alpha_2}\chi_2(0,t) + \frac{1}{\alpha_3}\chi_3(0,t).$$

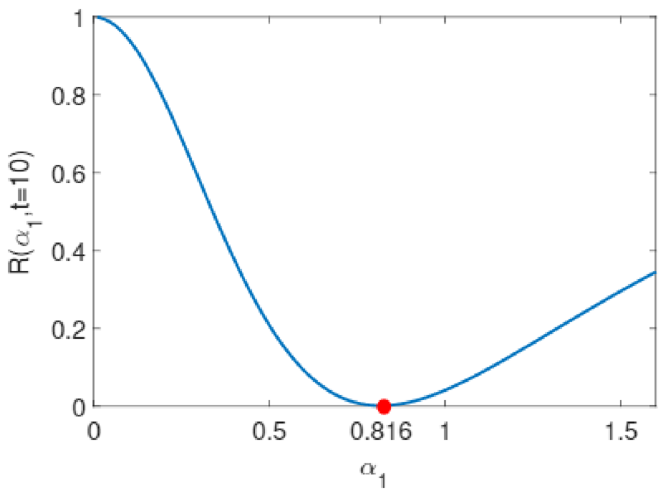
J. R. Yusupov, K. K. Sabirov, Q.U. Asadov, M. Ehrhardt, and **DM** Phys.Rev. E, **101,** 062208 (2020)

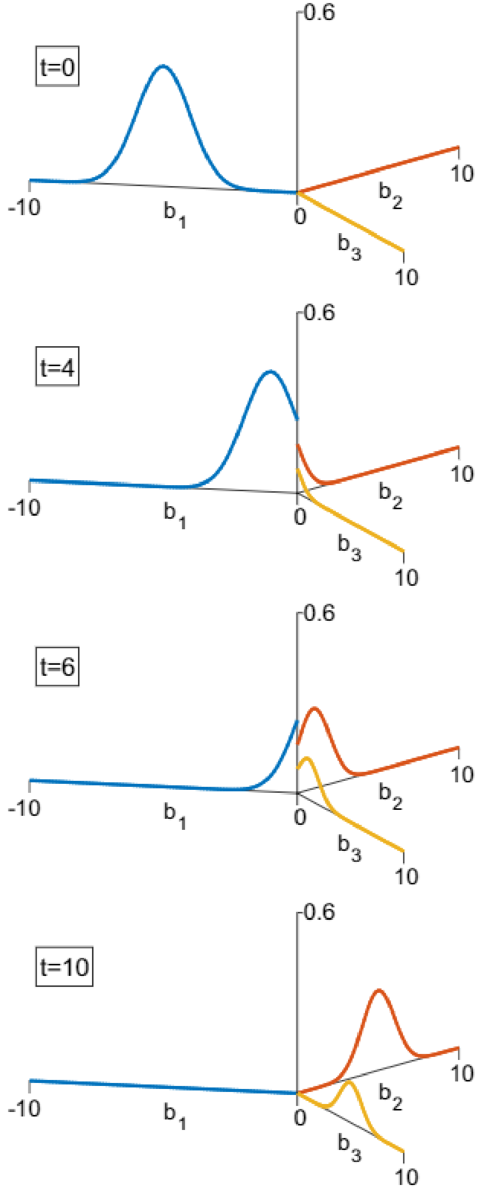
RELATIVISTIC QUASIPARTICLES IN TRANSPARENT QUANTUM GRAPHS

$$\chi_1(0,t) = A\left[\frac{d}{dt}\int_0^t I_0\big(m(t-\tau)\big)\phi_1(0,\tau)d\tau + im\int_0^t I_0\big(m(t-\tau)\big)\phi_1(0,\tau)d\tau\right],$$

where $A_1 = \alpha_1^2(\alpha_2^{-2} + \alpha_3^{-2})$ and $I_0(z)$ – Bessel functions

$$\frac{1}{\alpha_1^2} = \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2}.$$



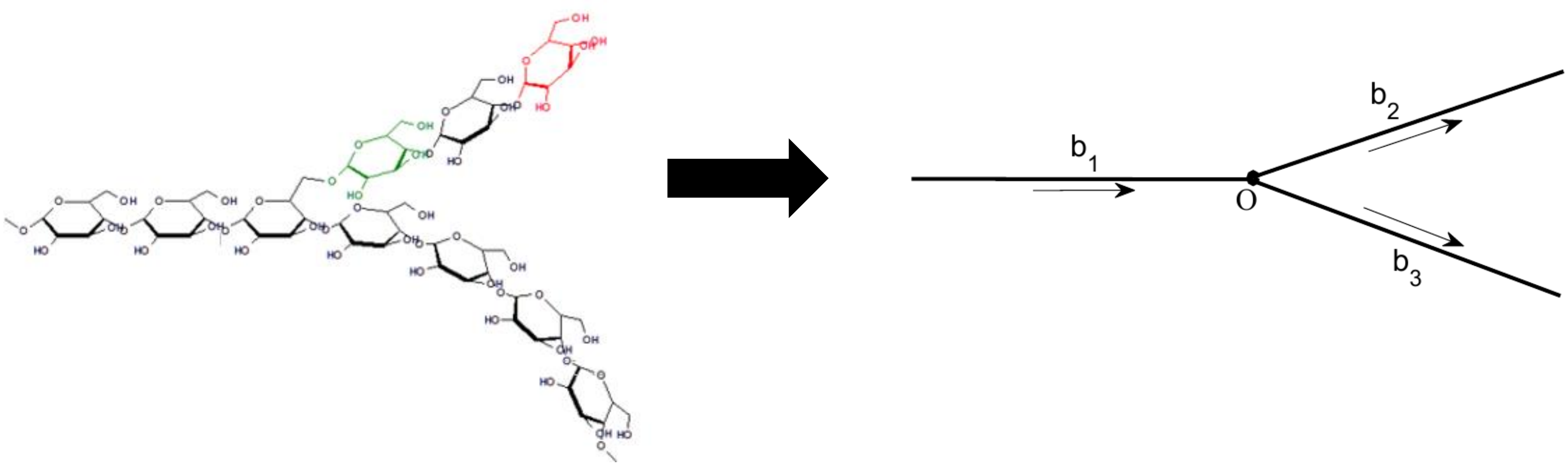


J. R. Yusupov, K. K. Sabirov, Q.U. Asadov, M. Ehrhardt, and **DM** Phys.Rev. E, **101,** 062208 (2020

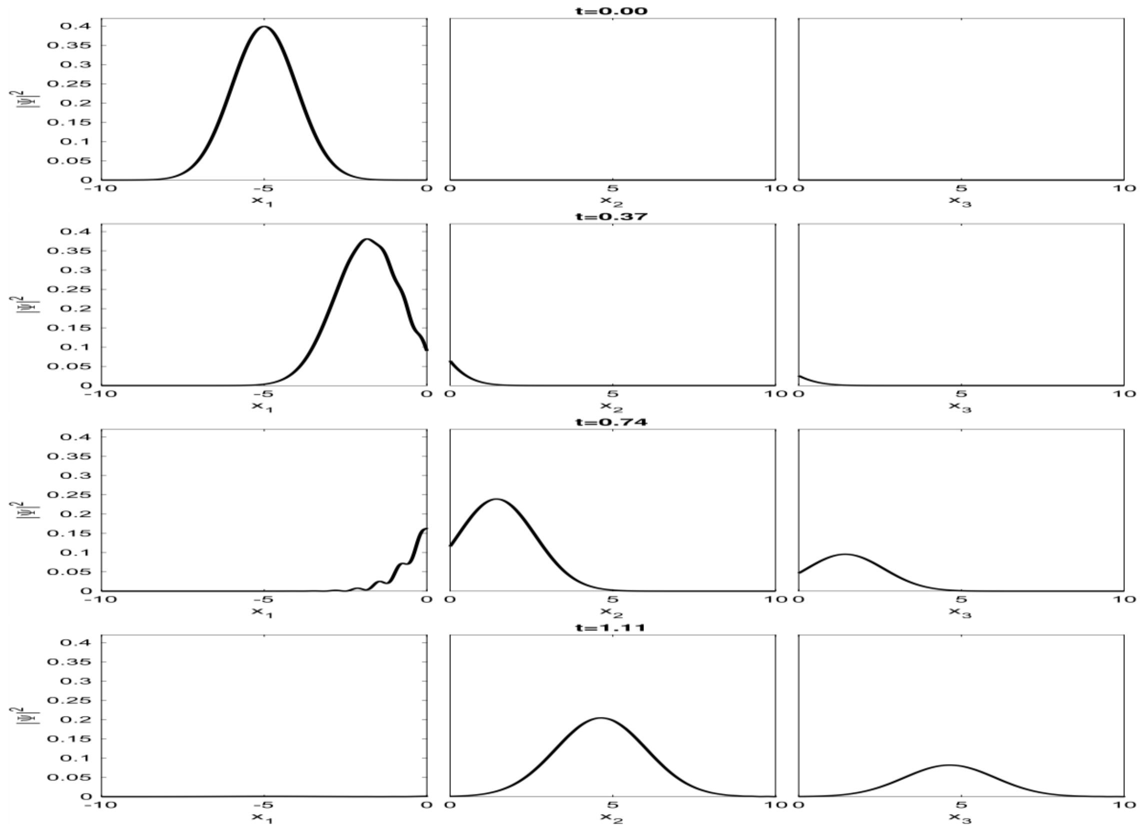
Dyamics ofexcitons in branched conducting polymers

Model: metric graph based approach

Branched polymer → Metric graph



Transport of excitons: Transmission through the branching point



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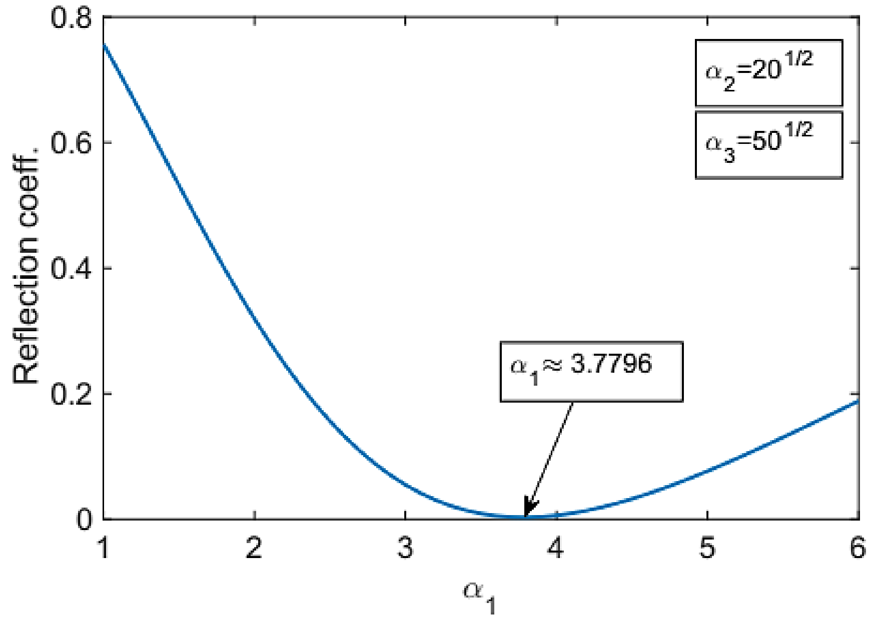
Transport of excitons: Transmission through the branching point



Exciton’s reflection coefficient at the polymer branched point.

Further progress made in modeling of transparent networks

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Summary

Basic theory for particle and wave dynamics in quantum networks is presented.

Theory of PT-symmetric graphs:

Breaking Hermitticity in quantum graphs

Experimental realization in microwave fibers

Transparent quantum graphs: Reflectionless transmission of waves through the vertices.

Outlook

Quantum teleportation on networks

Entangled quantum networks

Qubits in networks

Relativistic quantum graphs: Weyl and Majorana fermions in networks

Transparent microwave networks

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