J. Appl. Comput. Mech., 6(SI) (2020) 1426-1430
DOI: 10.22055/JACM.2020.34950.2548

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

# Larin Parameterization to Solve the Problem of Analytical Construction of the Optimal Regulator of Oscillatory Systems with Liquid Dampers

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Received September 16 2020; Revised November 10 2020; Accepted for publication November 11 2020.
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**Abstract.** The problem of the analytical construction of the optimal regulator of oscillatory systems with liquid dampers on the complex plane is considered. Since the fractional derivative is included in the differential equation describing the oscillatory systems with liquid dampers movement, the corresponding input-output transfer function also contains fractional rational orders, the general Larin parameterization scheme is modifying for this case. The results are illustrated by numerical examples and it is shown that they coincide with Letov's A.M. analytical construction of the optimal regulator.

**Keywords**: Optimal regulators, Larin parameterization, Frequency method, Transfer function, Oscillatory systems, Liquid damper, Fractional derivative.

## 1. Introduction

The problems of analytical construction of optimal regulators (ACOR) [1] play an important role in solving many practical problems, such as control flight problems [2, 3], vibration protection problems [4], control of nuclear reactors [5], as well as during construction optimal regulators in oil production [6], etc. In all these problems, the dynamics of the system is described by systems of classical ordinary differential equations. However, lately much attention has been paid to problems in which the movement of an object is described by a system whose equations include fractional derivatives in addition to ordinary derivatives [7].

Indeed, when a liquid damper enters a simple oscillatory system (OS), i.e. the mass of the OS moves inside the Newtonian fluid (see Fig. 1), then the mathematical model of this process is described by a second-order differential equation, which also includes the fractional derivative in [8, 10]. Naturally, for these systems, its optimal stabilization is relevant. Indeed, during oil production by the rod- pumping method , the plunger moves in a Newtonian fluid , and it is very important to stabilize it [10] in the neighborhood of corresponding programm trajectories and controls [11-15].

Recently, using ACOR [1], this problem is solved in the time domain, i.e. an optimal regulator is constructed that gives the closed-loop system asymptotic stability [15]. An example of this may be the case when the valve of the plunger does not work [10, 16] and the mass of the plunger moves idle, i.e. there is a constant volume of fluid inside the plunger.

The ACOR analogue for solving this problem in many cases can be limited, for example, when there are no restrictions on the controls in the quadratic quality criterion. In addition, it is necessary to bring the sought equation to a normal system, and this can increase the system dimension much more [15] . Therefore, for the solution of this problem it is possible to use ACOR , the frequency methods of the synthesis problem of optimal systems [17-26], etc. However, among them, the most common is the frequency method for solving the synthesis problem in the complex region [4, 17], which, in particular, is applied to stabilize the vibration protection system [4]. This method was further developed in a more general form [18–20] and in [6, 19, 20] it was shown that the remaining methods [20–25] are obtained from the results of [17–22] as special cases. Further, we will call this method as Larin frequency method to solve the problem of synthesis of optimal systems and will apply it to solution of the stabilization problem [27] of oscillatory systems with liquid dampers [8, 9], i.e. the case of fractional-derivative order systems [28-31].

In this paper, the problems of synthesis of optimal regulators for the simplest oscillatory systems [32,33] with liquid dampers are posed. After the statement of the problem for the equation of motion, using the Laplace transform, the quadratic functional using the Fourier transform and the Parseval identity [34] proceeds to the corresponding problem on the complex plane.

The case is considered when fractional orders are present in the denominator of the corresponding transfer function and VB Larin's method is generalized [17-19] for the given case when the numerator and denominator are odd numbers. Also, when one of them is even, the regularization [15] of similar problems is constructed, i.e. any accuracy can be approximated by the order of the fractional derivative with such a fraction whose numerator and denominator are odd numbers. The results are illustrated by numerical examples and a specific form of optimal regulators is given.

Published online: November 12 2020

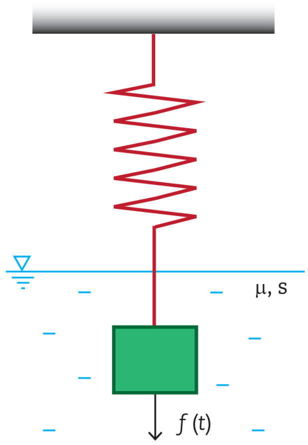


**Fig. 1.** Schematic representation of the problem.

## 2. Problem Statement

### 2.1. Time domain

Let the motion of an oscillatory system with liquid dampers [8] be described by the second order ordinary differential equation, along with the usual having a fractional derivative, in the form:

$$y''(x) + aD^{\alpha}y(x) + by(x) = u \tag{1a}$$

with initial conditions

$$y(0) = 0,\ y'(0) = y_1, \tag{1b}$$

where $a = 2S\sqrt{\mu\rho}/m$, $b = k/m$ and a rigid plate with a mass $m$ and area $S$ is considered, constant $\rho$ is fluid density, $\mu$ is viscosity of elasticity, constant $k$ characterizes the spring properties (see Fig.1).

The problem is to find such linear control law

$$u = Ky\,, \tag{2}$$

which would minimize the quadratic functional

$$J = \frac{1}{2}\int_0^{\infty}(ry^2 + Cu^2)dt \tag{3}$$

in the condition of the asymptotic stability of a closed system (1)+(2). As is shown in [15] from (2) $K$ is operator polynomial, which depends on $D^{(2q-1)/q}$ and the solution of the problem ACOR (1)-(2) in the time domain requires the reduction of equation (1) to a normal system with a step $1/q(\alpha = p/q)$, $p$ and $q$ are odd natural numbers, $\alpha \in (0,1)\cup(1,2)$.

### 2.2. Frequency domain

Now we will act differently, i.e. let us consider the solution of the AKOR problem (1) - (3) in the frequency domain. To do this, we apply the Laplace transform to equation (1). Then equation (1) takes the form

$$P(s)\tilde{y}(s) = M(s)\tilde{u}(s) + \psi(s), \tag{4}$$

where $\tilde{y}(s)$ and $\tilde{u}(s)$ are Laplace transform of a function $y(t)$ and $u(t)$, accordingly,

$$P(s) = s^2 + as^{p/q} + b,\ M = 1,\ \psi(s) = y_1\,. \tag{4'}$$

If for functional (3), we use the Fourier transform and apply the Parseval identity [32], then we have

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(r\tilde{y}(s)\tilde{y}(-s) + c\tilde{u}(s)\tilde{u}(-s))ds\,. \tag{5}$$

Then the task will be to find such law of regulation

$$\omega_0(s)\tilde{u}(s) = \omega_1(s)\tilde{y}(s) \tag{6}$$

so that the closed-loop system (4) + (6) would be asymptotically stable, and the functional (5) would take a minimum value.

## 3. LarinV.B. Parameterization [17]

As in [17-21], we construct a matrix $Z$

$$Z = \begin{bmatrix} P(s) & -M(s) \\ \alpha(s) & \beta(s) \end{bmatrix}, \tag{7}$$

where we have to choose the polynoms of parameters $\alpha(s)$ and $\beta(s)$, so that, $Z^{-1}$ was analytic on the right half-plane. For this $\det Z = P(s)\beta(s) + M(s)\alpha(s)$ must be as Gurvich, or constant, i.e.in this case, it suffices to choose

$$\beta(s) = 0\ ,\quad \alpha(s) = 1\ . \tag{8}$$

Using the Larin parameterization [17-19] we can easily show that $\omega(s) = \omega_1(s)/\omega_0(s)$ is defined as follows

$$\omega(s) = \frac{\Phi(s)P(s) - 1}{\Phi(s)}, \tag{9}$$

where $\Phi(s)$ is Larin parameter, which is physically realizable-analytical on the right half-plane [17]

$$\Phi(s) = -\frac{B_0(s)}{D(s)}, \tag{10}$$

but

$$\begin{gathered} B_0(s) + B_-(s) = \frac{T(s)}{D(-s)}, \\ D(s)D(-s) = (r + cP(s)P(-s))\psi_1^2 \\ T(s) = -cP(-s)\psi_1^2\ . \end{gathered} \tag{11}$$

Here $B_0(s)$ is an integer part, fractional part of the function $B_-(s)$ has poles in the right half-plane after separation of the expression $T(s)/D(-s)$, $D(s)$ has zeros in the left half-plane after factorization of the expression (11). Substituting $\Phi(s)$ from (10) into (9), for the feedback circuit coefficient $\omega(s)$, we have the following expression

$$\omega(s) = \frac{-\dfrac{B_0(s)}{D(s)}P(s) - 1}{-\dfrac{B_0(s)}{D(s)}} = \frac{B_0(s)P(s) + D(s)}{B_0(s)} \tag{12}$$

Thus from (13)

$$\omega_0(s) = B_0(s),\ \omega_1(s) = B_0(s)P(s) + D(s) \tag{13}$$

Now we show that the closed-loop system (4), (6) and (13) is asymptotically stable. We compose the determinant of the coefficients (4), (6) and take into account (13)

$$\begin{aligned} \det\begin{bmatrix} P(s) & -M(s) \\ \omega_1(s) & -\omega_0(s) \end{bmatrix} &= -P(s)\omega_0(s) + M(s)\,\omega_1(s) \\ &= -P(s)B_0(s) + P(s)B_0(s) + D(s) = D(s), \end{aligned}$$

i.e., the closed system is asymptotically stable. Another parameterization, the so-called Youla-Kucera-Desoer [23-25], unlike parameterization [17-19], suggests choosing $\alpha(s)$ and $\beta(s)$ from the following Diophantine equation

$$P(s)\beta(s) + M\alpha(s) = 1\ , \tag{14}$$

which is a special case of LarinV.B. parameterization [17,18,21, 22, 26]. Note that (8) also satisfies the Diaphont equation (14). And this is due to the fact that (14) is a special case of Gurvich from (7), i.e. the results of [23–25] are obtained as a special case of [17–19]. We illustrate the above with the following specific example.

## 4. Example

Let in (1)

$$a = 3,\ b = 1,\ \alpha = \frac{1}{3},\ r = 1,\ c = 1,\ \psi_1 = 1.$$

Then from (4')

$$P(s) = s^2 + 3s^{1/3} + 1,\ M = 1 \tag{Ex.1}$$

and from (10)

$$T(s) = -s^2 + 3s^{1/3} - 1 \tag{Ex.2}$$

$$D(s)D(-s) = 1 + (s^2 + 3s^{1/3} + 1)(s^2 - 3s^{1/3} + 1) = s^4 + 2s^2 - 9s^{2/3} + 2 \tag{Ex.3}$$

Now we write (Ex.3) with a step $1/3$ in the following form:

$$D(s)D(-s) = (s^{1/3})^{12} + 2(s^{1/3})^6 - 9(s^{1/3})^2 + 2 \tag{Ex.4}$$

We factorize (Ex.4) using [21,22] and for $D(s)$, we have the following expression

$$D(s) = (s^{1/3})^6 + 4.38092(s^{1/3})^5 + 9.5964(s^{1/3})^4 + 13.2864s + 12.162s^{1/3} + 6.5878s^{1/3} + 1.4142\ ,$$

whose zeros are in the left half-plane. Then from (10) we calculate $B_0(s)$ which is equal to $-1$, i.e. $B_0(s) = -1$.
Thus, the equation of regulator on the complex plane (6), (13) has the form

$$\tilde{u}(s) = (-0.4142 - 3.5878s^{1/3} - 12.162s^{2/3} - 13.2864s - 9.5964s^{4/3} - 4.3809s^{5/3})\,\tilde{y}(s) \tag{Ex.5}$$

and in the time domain will be

$$u(t) = -0.4142y(t) - 3.5878D^{1/3}y(t) - 12.162D^{2/3}y(t) - 13.2864Dy(t) - 9.5964D^{4/3}y(t) - 4.3809D^{5/3}y(t). \tag{Ex.6}$$

Note that the result (Ex.6) completely coincides with the result of [15] and it is shown there that the closed-loop system (1), (Ex.1), (Ex.6) is asymptotically stable.

## 5. Conclusion

The Larin parameterization is generalized to solve the linear-quadratic optimization problem of oscillatory systems with liquid dampers on the complex plane. Such an approach will allow us to extend this result to the general case, also the Larin parameterization can be used to stabilize the rod - pumping unit in oil production.

## Author Contributions

Aliev Fikret considered a general formulation of the problem and participated in obtaining the formula for the regulator coefficient. Aliev Nihan took part in obtaining results with fractional derivatives. Safarova Nargiz carried out all the computational procedures to obtain the equation (Ex.5) and (Ex.6). Velieva Naila carried out a computational procedure for factorization and separation (11) and (4). Also carried out the calculation programs. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

## Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

## Funding

The authors of this paper were financially supported from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 873071.

## Nomenclature

| | | | |
|---|---|---|---|
| $a$ | $= 2S\sqrt{\mu\rho}\,/\,m$ | $\rho$ | Fluid density [kg/m$^3$] |
| $b$ | $= k\,/\,m$ | $\mu$ | Viscosity of elasticity, constant |
| $m$ | Mass of the rigid plate [kg] | $k$ | Characterizes the spring properties |
| $S$ | Area of the rigid plate [m$^2$] | | |

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How to cite this article: Aliev F.A., Aliev N.A., Velieva N.I., Safarova N.A. Larin Parameterization to Solve the Problem of Analytical Construction of the Optimal Regulator of Oscillatory Systems with Liquid Dampers, *J. Appl. Comput. Mech.*, 6(SI), 2020, 1426–1430. https://doi.org/10.22055/JACM.2020.34950.2548