Appl. Comput. Math., V.20, N.3, 2021, pp.339-365

SOME MATHEMATICAL PROBLEMS AND THEIR SOLUTIONS FOR THE OSCILLATING SYSTEMS WITH LIQUID DAMPERS: A REVIEW

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Abstract. The mathematical problem of an oscillating system with liquid dampers is considered, such as finding the order of the fractional derivative of a subordinate term based on the given statistical data from practice, constructing. A solution of the corresponding system with nonseparated boundary conditions, is constructed including for large values of the head mass, finding asymptotic solutions on the first approximations, and constructing optimal regulators to stabilize the system around the corresponding program trajectories and controls.

Keywords: Oscillating Systems, Liquid Damper, Fractional Derivative, Asymptotic Representation, Optimal Control, Regulator, Boundary Conditions, Larin Parameterization, Discretization.

AMS Subject Classification: 33E12, 34A08, 44A10, 47J25, 65K10.

1. Introduction

The differential equations [24, 29, 30, 48] of the classical oscillating system

$$m\ddot{y}(t) + a\dot{y}(t) + by(t) = f(t), \tag{1}$$

play an important role in solving many problems of control [10], optimization [11, 16, 55], oil production of sucker-rod pumping units [32, 52, 67], etc., where m is the mass of the head, a and b are given real numbers with concrete physical significance, $f(t)$ is an external disturbance, a continuous real-valued function.

If the head of the oscillating system moves inside the Newtonian fluid, then the equation (1) ceases to describe the resulting process exactly and (1) is reduced to the next equation of the fractional derivative in the subordinate term [3, 5, 13, 54, 65, 72]:

$$m\ddot{y}(t) + aD^{\alpha}y(t) + by(t) = f(t), \ \ t \geq t_0 > 0, \alpha \in (0,1)\bigcup(1,2). \tag{2}$$

For equation (2), various problems can be considered:

(1) Determining the order, which is the fractional derivative of the subordinate term, using statistical data from practice (e.g., in oil production with a sucker rod pump unit, statistical data can be taken from the volume of the flow rate at different points in time).

*The authors of this paper were financially supported from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 873071.

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Manuscript received 21 September 2021.

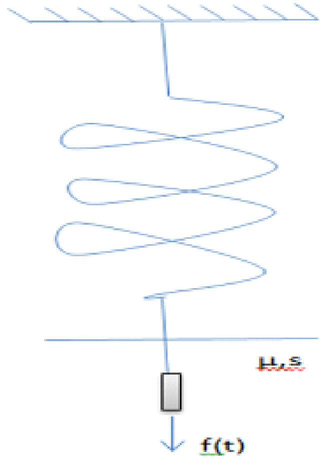

Figure 1.

(2) Solution of a non-local boundary value problem for equation (2).
(3) For sufficiently large values of m, finding the asymptotic representation by the first approximation.
    Note that the results of item 2 and item 3 can be successfully used to find program trajectories and controls for oil delivery by a sucker-rod pumping unit [8].
(4) Finding the optimal controllers to stabilize the motion of the head mass m around the corresponding program trajectories and controls. Here the controllers can be designed using time-frequency methods.

In our opinion, the second method is more acceptable for developing efficient computational methods.

Note that the solution of the problem given in items 1-4 was considered in the papers [13,14,45,48] for simple cases when m, a, b, α are constant real numbers. In the case of the motion of the head of mass m in the Newtonian fluid (in the case of oil production, the motion of the plunger in the fluid) is a function of t and as a consequence, after division by m, the coefficients of the obtained two terms depend on t. Therefore, the above methods are complicated and it is necessary to take into account the periodic problems [45] of choice of program trajectories and control, as well as their optimal stabilization. In this case, the determination of the fractional derivative becomes rather complicated and requires the discretization of the corresponding equation (2) [14]. Furthermore, the problem of selecting program trajectories, controls and optimal stabilization is posed for the periodic system where m and y operate according to this principle. This approach is also acceptable for the fact that the motion of the plunger in m and y is described by either differential or finite difference equations, which complicates the development of effective computational methods due to the inhomogeneity of the problem structures. Therefore, considering the discrete case facilitates the construction of computational methods [64] for the homogeneous-discrete case.

2. Stationary case. Determining the order of the fractional derivative α

2.1. Determining α using a discretized Volterra integral equation of the second kind.

Let the oscillating system with liquid dampers (2) with the initial conditions is given

$$y(t_0) = 0, \quad y'(t_0) = y_{10}. \tag{3}$$

Thus, we get the Cauchy problem (2), (3).

Taking into account the definitions of the fractional Riemann-Liouville derivative [27], the subordinate term in (2) takes the form:

$$D^{\alpha}y(t) = \frac{d^2}{dt^2}\int\limits_{t_0}^{t}\frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y(\tau)d\tau, \quad t \geq t_0 > 0, \;\; \alpha \in (1,2). \tag{4}$$

Then from (4) and integrating twice (2), we have:

$$y(t) + \int\limits_{t_0}^{t} K_{\alpha}(t-\tau)y(\tau)d\tau = F(t), \;\; t \geq t_0 \; > 0, \;\; \alpha \in (1,\, 2), \tag{5}$$

$$K_{\alpha}(t-\tau) = \frac{a}{m}\frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!} + \frac{b}{m}(t-\tau),$$

$$F(t) = \frac{1}{m}\int\limits_{t_0}^{t}(t-\tau)f(\tau)d\tau + y_{10}(t-t_0),$$

Now, discretizing the Volterra equations of the second kind (5) with a step $h = \frac{l-t_0}{n}$, we obtain:

$$y_i + \sum_{k=0}^{i-1} K_{\alpha}(t_i - t_k)y_k h = F_i, \quad i = \overline{1,n}, \tag{6}$$

where

$$y_i = y(t_i), \;\; t_i = t_0 + ih, \;\; \overline{i = 1,n}, \;\; t_n = t_0 + nh = t_0 + n\frac{l-t_0}{n} = l,$$

$$K_{\alpha}(t_i - t_k) = \frac{a}{m}\frac{(t_i - t_k)^{1-\alpha}}{(1-\alpha)!} + \frac{b}{m}(t_i - t_k), \quad k = \overline{0, i-1}, \;\; i = \overline{1,n},$$

$$F_i = \frac{1}{m}\sum_{k=0}^{i-1}(t_i - t_k)f(t_k)h + y_{10}(t_i - t_0), \quad i = \overline{1,n}.$$

Choosing α from $(0,\, 1)\bigcup(1,\; 2)$(the choice is given on the interval (1,2)) with a step $\frac{1}{q}$, we have:

$$\alpha_s = 1 + \frac{s}{q}, \;\; s = \overline{1, p-1}.$$

For the corresponding y_i from (6), we get:

$$y_i^s = F_i - h\sum_{m=0}^{i-1} K_{\alpha_s}(t_i - t_m)y_m^s, \;\; s = \overline{1, p-1}, \;\; i = \overline{1,n}.$$

To find the parameter α, we use the least squares method [8] and compose the following functional:

$$J = \sum_{i=1}^{n}\sum_{s=1}^{p-1}(y_i - y_i^s)^2,$$

where y_i depend on α (in the form (6)), and y_i^s are statistical data taking from practice. Then for determining α we get the equation:

$$\frac{\partial J}{\partial \alpha} = 2\sum_{i=1}^{n}\sum_{s=1}^{p-1}(y_i - y_i^s)\frac{\partial y_i}{\partial \alpha} = -2\sum_{i=1}^{n}\sum_{s=1}^{p-1}\left\{\left[-\sum_{k=0}^{i-1}K_\alpha(t_i - t_k)y_k h + F_i\right] - y_i^s\right\}\times$$
$$\times\sum_{p=0}^{i-1}\frac{\partial K_\alpha(t_i - t_p)}{\partial \alpha}y_p h = 0, \tag{7}$$

more simple form

$$+2\sum_{i=1}^{n}\sum_{s=1}^{p-1}\left[-\sum_{k=0}^{i-1}K_\alpha(t_i - t_k)y_k h + F_i - y_i^s\right]\times$$
$$\times\sum_{p=0}^{i-1}\frac{1}{\Gamma^2(2-\alpha)}\left[\frac{a}{m}(t_i - t_p)^{1-\alpha}\ln(t_i - t_p)\Gamma(2-\alpha)-\right. \tag{8}$$
$$\left.-\frac{a}{m}(t_i - t_p)^{\ 1-\alpha}\int_0^\infty e^{-t}t^{1-\alpha}\ln t dt\right]y_p h = 0.$$

The transcendental equation (8) has solved with respect to α in one or another way, we determine the desired fractional derivative, where $\Gamma^2(2-\alpha)$ is the Euler function.

2.2. **Computational algorithm.** Let us present the following algorithm for the solution of transcendental equation (8):

Algorithm:

(1) ε - determining the accuracy of the solution of the problem and parameters a, b, F_i are set.
(2) The segment $[\alpha_1, \alpha_2]$ is defined, where the root of the function $\frac{\partial J(\alpha)}{\partial \alpha}$ is sought, where $\frac{\partial J(\alpha_1)}{\partial \alpha}\cdot\frac{\partial J(\alpha_2)}{\partial \alpha} < 0$
(3) $N = 10$, $i = 4$ are given.
(4) The values of statistical data are formed from practice.
(5) Calculate the midpoint $\alpha = \frac{\alpha_1+\alpha_2}{2}$ and the value of $\frac{\partial J(\alpha)}{\partial \alpha}$ according to the formula (7).
(6) If $\left|\frac{\partial J(\alpha)}{\partial \alpha}\right| < \varepsilon$ the process stops. Otherwise, if $\frac{\partial J(\alpha_1)}{\partial \alpha}\frac{\partial J(\alpha)}{\partial \alpha} < 0$, we denote $\alpha_2 = \alpha$ and if $\frac{\partial J(\alpha)}{\partial \alpha}\frac{\partial J(\alpha_2)}{\partial \alpha} < 0$, then $\alpha_1 = \alpha$, go to step 5.

Note that by setting $n = \frac{\alpha_2-\alpha_1}{N}$ we calculate $\alpha_i = \alpha_1 + (i-1)h$ for $i = \overline{1, N+1}$ and define the values of the function $\frac{\partial J(\alpha)}{\partial \alpha}$ whose graph is shown in Fig.2.

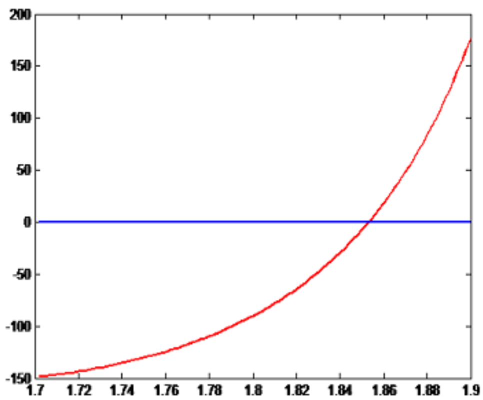

Figure 2. Dependency graph $\frac{\partial J(\alpha)}{\partial \alpha}$ on α

Thus, the performed calculation [15] shows the plausibility of the obtained results. For sufficiently large m, we can obtain an asymptotic representation [56] of α in terms $\varepsilon = \frac{1}{m}$.

3. Method for solving oscillatory systems, where fractional derivatives with a step of $1/q$ $(q \in N)$ entering both the equation of motion and nonlocal boundary conditions

3.1. **The general case.** The general form of the equation (2) has the form [22,61]:

$$y''(t) + \sum_{k=0}^{2q-1} a_k D^{\frac{k}{q}} y(t) = f(t), \qquad 0 < t_0 < t < l, \tag{9}$$

where $q \in N = \{1, 2, 3, ...\}$- natural numbers, $a_k \in R$- given real numbers, $f(t)$ – continuous real valued function, in equation (9), the order of the derivative varies by $1/q$ to the second order.

For equation (9), we present the following nonlocal boundary conditions:

$$\sum_{k=0}^{2q-1} \left[\alpha_{jk}\, D^{\frac{k}{q}} y(t)\Big|_{t=t_0} + \beta_{jk} D^{\frac{k}{q}} y(t)\Big|_{t=l} \right] = \gamma_j, \qquad j = \overline{1,\ 2q}, \tag{10}$$

where α_{jk}, β_{jk} and γ_j are given real numbers and boundary conditions (10) are linearly independent.

By means of substituting

$$D^{\frac{k}{q}} y(t) = z_k(t), \quad k = \overline{0, 2q-1},$$

$$D^{\frac{2q}{q}} y(t) \equiv y''(t) = D^{\frac{1}{q}} z_{2q-1}(t), \tag{11}$$

the boundary value problem (10) is reduced to the form:

$$D^{\frac{1}{q}} z(t) = Az(t) + B(t), \qquad 0 < t_0 < t < l, \tag{12}$$

$$\theta_1 z(t_0) + \theta_2 z(l) = \gamma, \tag{13}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & . & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & . & . & . & . & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & . & . & . & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & . & . & . & . & 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 & -a_3 & -a_4 & . & . & . & . & -a_{2q-3} & -a_{2q-2} & -a_{2q-1} \end{pmatrix}, \tag{14}$$

$$B(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & . & . & . & . & . & 0 & 0 & 0 & f(t) \end{pmatrix}^T, \qquad 0 < t_0 < t < l, \tag{15}$$

$$z(t) = \begin{pmatrix} z_0(t) & z_1(t) & z_2(t) & ... & z_{2q-3}(t) & z_{2q-2}(t) & z_{2q-1}(t) \end{pmatrix}^T, \ 0 < t_0 < t < l, \tag{16}$$

$$\theta_1 = (\alpha_{jk})_{j=\overline{1,\ 2q},\, k=\overline{0,\ 2q-1}}, \quad \theta_2 = (\beta_{jk})_{j=\overline{1,\ 2q},\, k=\overline{0,\ 2q-1}}, \quad \gamma = (\gamma_1, \gamma_2, ..., \gamma_{2q})^T.$$

Let

$$T = (t_{ij})_{i,j=1}^{2q}$$

matrix transformation reduces A to a diagonal form, i.e.,

$$AT = T\hat{A},$$

where

$$\hat{A} = \begin{pmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \ddots & & \\ & & & \lambda_{2q-1} & \\ 0 & & & & \lambda_{2q} \end{pmatrix}.$$

Then with the replacement

$$z(t) = Tw(t),$$

the homogeneous equation corresponding to (12) is reduced to the form:

$$D^{\frac{1}{q}} w(t) = \hat{A} w(t),$$

the solution of which is given using the Mittag-Leffler function [2,54,66-69,70] has the following form:

$$w_k(t) = \sum_{m=0}^{\infty} \lambda_k^m \frac{t^{-1+\frac{m+1}{q}}}{\left(-1+\frac{m+1}{q}\right)!}, \quad k = \overline{1, 2q},$$

where the elements λ_k of the diagonal matrix $\hat{A}$ are determined from the equation

$$\det(A - \lambda E) = |A - \lambda E| = 0,$$

E - unit matrix of order $2q$. Thus, the matrix solution [20] of the homogeneous equation corresponding to (12) has the form:

$$Z(t) = \sum_{k=0}^{\infty} A^k \frac{t^{-1+\frac{k+1}{q}}}{\left(-1+\frac{k+1}{q}\right)!}.$$

For the general solution of the homogeneous equation corresponding to (12) we obtain:

$$z(t) = Z(t)\,C = \sum_{k=0}^{\infty} A^k \frac{t^{-1+\frac{k+1}{q}}}{\left(-1+\frac{k+1}{q}\right)!} C, \tag{17}$$

where C- arbitrary column with the size $2q$ with constant elements. If

$$f(t) = 0,$$

taking into account (17) in (13) we have:

$$\theta_1 Z(t_0)\,C + \theta_2 Z(l)\,C = \gamma,$$

or

$$C = [\theta_1 Z(t_0) + \theta_2 Z(l)]^{-1} \gamma, \tag{18}$$

if

$$\det[\theta_1 Z(t_0) + \theta_2 Z(l)] \neq 0. \tag{19}$$

Then we get

Theorem 1. If a_k, α_{jk}, β_{jk}, γ_j, $j = \overline{1,\ 2q}$, $k = \overline{0,\ 2q-1}$ are given parameters, $f(t) = 0$. Also the given parameters and the condition (10) are linearly independent, then the solution of the problem (12), (13) is given in the form (17), where C is defined in the form (18) under the condition (19).

Now we will consider the homogeneous case. By integrating the equation (12) of order $1/q$, we reduce to the following Volterra integral equation of the second kind [58,60,75]

$$z(t) = A\int\limits_{t_0}^{t} \frac{(t-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!} z(\tau)\, d\tau + F(t) + C\frac{t^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}, \tag{20}$$

$$\theta_1 z(t_0) + \theta_2 z(l) = \gamma,$$

where

$$F(t) = \int\limits_{t_0}^{t} \frac{(t-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!} B(\tau)\, d\tau.$$

Substituting (20) into (13), we have

$$\theta_1 C\frac{t_0^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!} + \theta_2\left[A\int\limits_{t_0}^{l} \frac{(l-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!} z(\tau)d\tau + F(l) + C\frac{l^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}\right] = \gamma, \tag{21}$$

and

$$C = \left(\theta_1\frac{t_0^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!} + \theta_2\frac{l^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}\right)^{-1}\left[\gamma - \theta_2\left(A\int\limits_{t_0}^{l} \frac{(l-\tau)^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!} z(\tau)d\tau + F(l)\right)\right] \tag{22}$$

with condition

$$\det\left(\theta_1\frac{t_0^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!} + \theta_2\frac{l^{\frac{1}{q}-1}}{\left(\frac{1}{q}-1\right)!}\right) \neq 0. \tag{23}$$

Substituting (22) into (20), we arrive at an integral equation containing both Volterra and Fredholm terms [17]. The theory of such equations was developed in the work [49].

3.2. **Asymptotic method.** In the previous problem, both sides of (9) was divided into m (for large values of m) and assumed that $\frac{1}{m} = \varepsilon$, then we have:

$$y''(x) + \sum_{k=0}^{2q-1} \varepsilon a_k D^{\frac{k}{q}} y(t) = \varepsilon f(t), \ \ 0 < t_0 < t < l,$$

$$\sum_{k=0}^{2q-1}\left[\alpha_{jk} D^{\frac{k}{q}}\ y(t)|_{t=t_0} + \beta_{jk} D^{\frac{k}{q}} y(t)|_{t=l}\right] = \gamma_j, \quad j = \overline{1,2q}.$$

Similar to the previous one in subsection 3.1., we arrive at the following equations:

$$D^{\frac{1}{q}} z(t,\varepsilon) = A(\varepsilon) z(t,\varepsilon) + B(t,\varepsilon), \tag{24}$$

$$\theta_1 z(t_0,\varepsilon) + \theta_2 z(l,\varepsilon) = \gamma, \tag{25}$$

where

$$z(t,\varepsilon) = (z_0(t,\varepsilon)\ z_1(t,\varepsilon)\ z_2(t,\varepsilon)\ ...\ z_{2q-1}(t,\varepsilon)\ )^T,$$

$$A(\varepsilon) = A_0 + \varepsilon A_1, \tag{26}$$

$$B(t,\varepsilon) = \varepsilon l_{2q} f(t), \tag{27}$$

$\alpha,\ \beta,\ \gamma$ are given in the previous case, A_0- nilpotent of order $2q$, A_1-zero matrix in the last 2q-th row $(-a_0\ -a_1\ -a_2\ ...\ -a_{2q-2}\ -a_{2q-1})$,$l_{2q} = (0\ 0\ 0...0\ 1)^T$ order $2q$.

$$A_0^{2q} = 0,\ A_1^k = (-1)^{k-1} a_{2q-1}^{k-1} A_1. \tag{28}$$

The solution of the system of differential equation (24) will be found in the form of the following series:

$$z(t,\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k z^{(k)}(t). \tag{29}$$

Substituting (29) into (24) taking into account (26)-(28) we get

$$D^{\frac{1}{q}} z^{(0)}(t) = A_0 z^{(0)}(t), \tag{30}$$

$$D^{\frac{1}{q}} z^{(1)}(t) = A_0 z^{(1)}(t) + (A_1 z^{(0)}(t) + l_{2q} f(t)), \tag{31}$$

$$D^{\frac{1}{q}} z^{(s)}(t) = A_0 z^{(s)}(t) + A_1 z^{(s-1)}(t), \quad s \geq 2$$

with boundary conditions

$$\theta_1 z^{(0)}(t_0) + \theta_2 z^{(0)}(l) = \gamma, \tag{32}$$

$$\theta_1 z^{(s)}(t_0) + \theta_2 z^{(s)}(l) = 0,\ \ s \geq 1. \tag{33}$$

Taking into account the condition (28) the matrix solution of system (30) has the form:

$$Z^0(t) = \sum_{k=0}^{2q-1} A_0^k \frac{t^{-1+\frac{k+1}{q}}}{(-1+\frac{k+1}{q})!},$$

and the solution of boundary problem (30), (32) is found as follows:

$$z^{(0)}(t) = Z^{(0)}(t) \left[\theta_1 Z^{(0)}(t_0) + \theta_2 Z^{(0)}(l)\right]^{-1} \gamma, \tag{34}$$

if

$$\det \left[\theta_1 Z^{(0)}(t_0) + \theta_2 Z^{(0)}(l)\right] \neq 0.$$

The solution of homogeneous equation corresponding to equations (31) has the form:

$$z^{(1)}(t) = Z^{(0)}(t)C,$$

where C-vector column of length $2q$ with arbitrary constant elements, and the solution of the boundary value problem (31), (33) for $s = 1$ has the form:

$$z^{(1)}(t) = -Z^{(0)}(t)[\theta_1 Z^{(0)}(t_0)+\theta_2 Z^{(0)}(l)]^{-1}\left\{\theta_2\begin{pmatrix}0\\0\\\vdots\\0\\\tilde{A}(l)\end{pmatrix}+\theta_1\begin{pmatrix}0\\0\\\vdots\\0\\\tilde{A}(t_0)\end{pmatrix}\right\}+\begin{pmatrix}0\\0\\\vdots\\0\\\tilde{A}(t)\end{pmatrix}, \quad (35)$$

where

$$\tilde{A}(t) = I^{\frac{1}{q}}\left\{f(t) + A_{1,2q}Z^{(0)}(t)[\theta_1 Z^{(0)}(t_0) + \theta_2 Z^{(0)}(l)]^{-1}\gamma\right\}. \quad (36)$$

If we restrict with zero and first approximations, then for the solution of the boundary value problem (24), (25) we obtain:

$$\tilde{z}(t) = z^{(0)}(t) + \varepsilon z^{(1)}(t), \quad (37)$$

where $z^{(0)}(t)$ and $z^{(1)}(t)$ are defined in (34), (35) correspondingly.

Note that the results presented in item 3 can be used to construct program trajectories and control oscillatory systems with liquid dampers [56].

3.3. **Fractional Oscillating Systems.** The following Riemann–Liouville linear fractional differential delay equation of order $1 < 2\alpha \leq 2$ with an initial condition with a singularity is studied in [19]:

$$\begin{cases} \left\{D^{\alpha}_{-h^+}\left(D^{\alpha}_{-h^+}x\right)(t) + A^2 x(t) + \Omega^2 x(t-h) = f(t), \quad t \geq 0,\ h > 0,\right. \\ x(t) = \phi(t), \ \left(D^{\alpha}_{-h^+}x\right)(t) = \left(D^{\alpha}_{-h^+}\phi\right)(t), \quad -h \leq t \leq 0, \\ \left(I^{1-\alpha}_{-h^+}x\right)(-h^+) = \phi(-h), \ \left(I^{1-\alpha}_{-h^+}\left(D^{\alpha}_{-h^+}x\right)\right)(-h^+) = D^{\alpha}_{-h^+}\phi(-h), \end{cases}$$

where $D^{\alpha}_{-h^+}$ stands for the Riemann-Liouville fractional derivative of order $\frac{1}{2} < \alpha \leq 1$ with lower limit $-h$, $A, \Omega \in R^n$, $f \in C([0,\infty), R^n)$, $\phi \in C^1([-h,0], R^n)$.

We introduce the concept of a fractional delayed matrix cosine and sine:

Definition 1. [46] The matrix function

$$\cos_{h,\alpha,\beta}\{A,\Omega;t\} = \begin{cases} \Theta, \quad -\infty < t < -h, \\ \sum\limits_{p=0}^{\infty}(-1)^p A^{2p}\frac{(t+h)^{2p\alpha+\beta-1}}{\Gamma(2p\alpha+\beta)}, \quad -h \leq t < 0, \\ \sum\limits_{p=0}^{\infty}(-1)^p A^{2p}\frac{(t+h)^{2p\alpha+\beta-1}}{\Gamma(2p\alpha+\beta)} \\ +\Omega^2\sum\limits_{p=1}^{\infty}\begin{pmatrix}p\\1\end{pmatrix}(-1)^p A^{2p-2}\frac{t^{2p\alpha+\beta-1}}{\Gamma(2p\alpha+\beta)}, \quad 0 \leq t < h, \\ \vdots \\ \sum\limits_{p=0}^{\infty}(-1)^p A^{2p}\frac{(t+h)^{2p\alpha+\beta-1}}{\Gamma(2p\alpha+\beta)} \\ +\cdots+\Omega^{2k}\sum\limits_{p=k}^{\infty}\begin{pmatrix}p\\k\end{pmatrix}(-1)^p A^{2p-2k}\frac{(t-(k-1)h)^{2p\alpha+\beta-1}}{\Gamma(2p\alpha+\beta)}, \quad (k-1)h \leq t < kh \end{cases}$$

is called the fractional delayed matrix cosine of two parameters.

Definition 2. [46] The matrix function

$$\sin_{h,\alpha,\beta}\{A,\Omega;t\} = \begin{cases} \Theta, \quad -\infty < t < -h, \\ \sum\limits_{p=0}^{\infty}(-1)^p A^{2p}\frac{(t+h)^{(2p+1)\alpha+\beta-1}}{\Gamma((2p+1)\alpha+\beta)}, \quad -h \le t < 0, \\ \sum\limits_{p=0}^{\infty}(-1)^p A^{2p}\frac{(t+h)^{(2p+1)\alpha+\beta-1}}{\Gamma((2p+1)\alpha+\beta)} \\ +\Omega^2\sum\limits_{p=1}^{\infty}\begin{pmatrix} p \\ 1 \end{pmatrix}(-1)^p A^{2p-2}\frac{t^{(2p+1)\alpha+\beta-1}}{\Gamma((2p+1)\alpha+\beta)}, \quad 0 \le t < h, \\ \vdots \\ \sum\limits_{p=0}^{\infty}(-1)^p A^{2p}\frac{(t+h)^{(2p+1)\alpha+\beta-1}}{\Gamma((2p+1)\alpha+\beta)} \\ +\cdots+\Omega^{2k}\sum\limits_{p=k}^{\infty}\begin{pmatrix} p \\ k \end{pmatrix}(-1)^p A^{2p-2k}\frac{(t-(k-1)h)^{(2p+1)\alpha+\beta-1}}{\Gamma((2p+1)\alpha+\beta)}, \quad (k-1)h \le t < kh \end{cases}$$

is called the fractional delayed matrix sine of two parameters.

We use the variation of constants method to obtain an explicit formula for a solution to the initial value problem for a Riemann-Liouville sequential fractional linear time-delay system of order $1 < 2\alpha \le 2$. Our results extend those for fractional linear time-delay systems and novel for the classical case $\alpha = 1$.

Theorem 2. [46] Let A and Ω be a permutable matrices. A solution $x(t) \in C^1([-h,0],R^n) \cap C^2([0,\infty),R^n)$ of (36) has the form:

$$\begin{aligned} x(t) &= \phi(-h)\cos_{h,\alpha,\alpha}\{A,\Omega;t\} + D^{\alpha}_{-h^+}\phi(-h)\sin_{h,\alpha,\alpha}\{A,\Omega;t\} \\ &+ \int\limits_{-h}^{0}\sin_{h,\alpha,\alpha}\{A,\Omega;t-h-s\}\left[\left(D^{\alpha}_{-h^+}\left(D^{\alpha}_{-h^+}\phi\right)\right)(s) + A^2\phi(s)\right]ds \\ &+ \int\limits_{0}^{t}\sin_{h,\alpha,\alpha}\{A,\Omega;t-h-s\}f(s)\,ds. \end{aligned}$$

FDEs containing not only one fractional derivative [2,3,5] but also more than one fractional derivative are intensively studied in many complex systems. Recently, the authors illustrated the physical processes with two essential mathematical ways: multi-term equations [19,44,63] and multi-order systems [31,33].

Multi-term fractional differential equations have been studied due to their applications in modeling and solved using various mathematical methods. In [44] have been solved the multi-term FDEs with constant coefficients and with the Liouville-Caputo fractional derivatives by using the method of operational calculus. Furthermore, in [19] have been considered the multi-term fractional relaxation equations with Liouville-Caputo fractional derivatives by using Laplace transform technique and studied the fundamental and impulse-response solutions of the initial value problem (IVP). Extension of the multi-order FDEs with variable coefficients in terms of generalized fractional derivatives have been investigated in [63].

As one of the important special cases of multi-term FDEs, Bagley-Torvik equations have been discussed by means of analytical methods in [28,47,59,73,76]. In [28] have been studied the general analytical solutions of the generalized Bagley-Torvik equations. An analytical approach on defense expenditure and economic growth with Liouville-Caputo fractional differentiation operator of order $0 < \alpha < 2$ by means of three-parameter Mittag-Leer functions using the Laplace integral transform method. In [73] have been reduced the following Bagley-Torvik equation

$$u''(t) + \mu\left({}^{C}D^{\alpha}_{0^+}u\right)(t) + u(t) = 0, \quad \mu > 0,$$

where $\alpha = 1/2$ or $\alpha = 2/3$, to the sequential FDEs and introduced a general solution of (37) by using the approach related to characteristic roots. In [28], Fazli and Nieto have investigated existence and uniqueness results and also approximations of the solutions to the Cauchy problem for the following Bagley-Torvik equation with fractional order of $1 < \alpha < 2$:

$$\begin{cases} \left(D^2 + A\,{}^{C}D^{\alpha} + B\right) u\left(t\right) = f\left(t\right), \quad 0 < t \leq 1, \\ u\left(0\right) = a, \quad u'\left(0\right) = b, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Liouville-Caputo fractional derivative, $f : [0,1] \to R$ is a given continuous function, a, b, A, B are real numbers. In [76] have been considered a general form of the Bagley-Torvik equation as follows:

$$\begin{cases} \lambda_2 \left({}^{C}D_{0+}^{\alpha} u\right)(t) + \lambda_1 \left({}^{C}D_{0+}^{\beta} u\right)(t) + \lambda_0 u\left(t\right) = f\left(t\right), \quad 0 < t \leq 1, \\ u\left(0\right) = a, \quad u'\left(0\right) = b, \end{cases}$$

and introduced analytical representation of solutions in terms of "generalized G-function" by using Laplace transform technique.

In the view of numerical methods have been applied to get approximate analytical solutions of the Bagley-Torvik equations like Adomian decomposition method [62], hybrid functions method [50], wavelet technique [71] and others. In [26] have been considered the discretization of Bagley-Torvik equations by means of fractional linear multistep methods and Adams type predictor-corrector pairs. In [70] have been investigated the solutions of a fractionally-damped generalized Bagley-Torvik equation whose damping characteristics are well-defined by means of the Riemann-Liouville and Liouville-Caputo types fractional differential operators via the homotopy analysis method which is implemented for computing the dynamic response analysis.

4. Construction of optimal regulators of stabilizing oscillatory systems with liquid dampers.

An algorithm is given for the design of optimal regulators for oscillatory systems using the method of liquid dampers by the analytical design of optimal Letov A.M. regulators [4] (this method being called the time method). Then one proceeds to the frequency method, i.e., the equation of motion by the Laplace transform and the Fourier transform, translates the quadratic functional into an algebraic equation, and then uses the elements of the calculus of variations to construct an optimal regulator. This method is called Larin parameterization to construct optimal regulators for linear quadratic control problems on an infinite time interval.

4.1. **Construction of optimal regulators by the Letov's method [4].** Let the motion of the object described by the equation (9) has the following initial conditions

$$D^{\frac{k}{q}} y(t)\Big|_{t=t_0} = y_k(t_0), \;\; k = \overline{0,\, 2q-1}, \tag{38}$$

where $y(t)$ is the control action, $y_k(t_0)$ are given real numbers.

After corresponding transformations (9) is reduced to the normal system (12), i.e.

$$D^{\frac{1}{q}} z(t) = Az(t) + Gu(t) \tag{39}$$

with initial conditions

$$z(t_0) = z_0 = (y_0(t_0),\; y_1(t_0),\; \dots\, , y_{2q-1}(t_0))^T,$$

where $z(\cdot),\; A,\; u(t) = B(t)$ are defined as (14)-(16), $u(t)$ is the control action and $G = (0\;0\;\dots\;0\;1)^T$.

The problem consist of finding the linear control law

$$u(t) = Kz(t), \tag{40}$$

so that the closed system (39), (40).

$$D^{\frac{1}{q}}z(t) = (A + GK)z(t) \tag{41}$$

be asymptotically stable and the next quadratic functional

$$J = \int_0^\infty (z'(t)Qz(t) + u'(t)Ru(t))dt$$

received its minimum value along solutions (41). Here the sought constant matrix K with the corresponding dimension, matrices $Q \geq 0$, $R > 0$ are given symmetric and also have the corresponding dimensions.

As is shown in [4], the desired feedback target matrix K has the form:

$$K = -R^{-1}\hat{T}_3\hat{T}_1^{-1}, \tag{42}$$

where

$$\hat{T}_1^{-1}H\hat{T} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix}, \quad H = \begin{bmatrix} A & -GR^{-1}G' \\ -Q & -A' \end{bmatrix}, \quad T = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 \\ \hat{T}_3 & \hat{T}_4 \end{bmatrix},$$

and H_+ with the dimension $q \times q$ has eigenvalues on the left half-plane, H_- - on the right half-plane. In this case, the closed system (38)

$$D^\alpha z(t) = (A - GR^{-1}G'\hat{T}_3\hat{T}_1^{-1})z(t), \;\; z(t_0) = z_0$$

has a solution

$$z(t) = T_1\tilde{X}_1(t)C_1,$$

which for $t \to \infty$, $\tilde{X}_1(t) \to 0$ and therefore $z(t) \to 0$, where is defined in [5]:

$$\hat{X}_1(t) = \sum_{s=0}^{p-2} A_+^{\frac{s+p}{q}} e^{A_+^{\frac{p}{q}}t} \frac{1}{\Gamma(\frac{s+1}{p})} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} A_+^{\frac{pk}{q}} \frac{(t-t_0)^{\frac{s+1}{p}+k}}{\left(\frac{s+1}{p}+k\right)!} +$$

$$+ \sum_{s=0}^{p-2} A_+^{\frac{s}{q}} e^{A_+^{\frac{p}{q}}t_0} \frac{t^{\frac{s+1}{p}-1}}{\Gamma(\frac{s+1}{p})} + A_+^{\frac{p-1}{q}} e^{A_+^{\frac{p}{q}}t},$$

C_1 is an arbitrary constant vector defining from the initial condition $z(t_0) = z_0$.

4.2. **Larin V.B. parameterization.** Consider a simple form of the equation (9), i.e. let

$$my''(t) + aD^\alpha y(t) + by(t) = u(t), \quad \alpha = \frac{p}{q}, \tag{43}$$

where α is one of $\frac{k}{q}$ in (9) and has an initial condition

$$y(t_0) = 0, \quad y'(t_0) = y_1.$$

The problem consists of finding

$$u = Ky \tag{44}$$

that the functional

$$J = \frac{1}{2}\int_0^\infty (ry^2 + cu^2)dt \tag{45}$$

obtains a minimum value and the closed system (43), (44) is asymptotically stable. Taking the Laplace transform to (43), we have

$$p(s)\tilde{y}(s) = M(s)\tilde{u}(s) + \psi(s), \tag{46}$$

where

$$p(s) = ms^2 + as^{\frac{p}{q}} + b, \quad M(s) = 1, \quad \psi(s) = y_1.$$

Using Fourier transforms and Parseval's identity [23], we write functional (45) in the form

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} (r\,\tilde{y}(s)\tilde{y}(-s) + c\tilde{u}(s)\tilde{u}(-s))ds. \tag{47}$$

In a complex area, it is necessary to find a law on regulation

$$\omega_0(s)\tilde{u}(s) = \omega_1(s)y(s), \tag{48}$$

so that the functional (47) receives the minimum value, and the closed system (46), (48) is asymptotically stable.

Composing the matrix Z from [10,11,38,43]

$$Z = \begin{bmatrix} P(s) & -M(s) \\ A(s) & B(s) \end{bmatrix},$$

we select varying parameters $A(s), B(s)$ so that $Z^{-1}(s)$ is analytically on the right half-plane, i.e. $\det Z(s) = P(s)B(s) + M(s)A(s)$ should be Hurwitz or constant. In this case

$$A(s) = 1, \quad B(s) = 0.$$

Using Larin parameterization [6,42,43] and accepting $\omega(s) = \frac{\omega_1(s)}{\omega_0(s)}$ we get

$$\omega(s) = \frac{\Phi(s)P(s) - 1}{\Phi(s)}, \tag{49}$$

where $\Phi(s)$ is Larin parameter [38] physically realizable - analytically on the right half-plane

$$\Phi(s) = \frac{B_0(s)}{D(s)}, \tag{50}$$

where

$$B_0(s) + B_+(s) = \frac{T(s)}{D(-s)}, \;\; D_+(s)D_-(s) = (r + cP(s)P(-s))K\psi(s).$$

$$T(s) = -CP(-s)\psi^2. \tag{51}$$

Here B_0 is the integer part, the fractional part $B_-(s)$ has poles in the right half-plane after separating the expressions $\frac{T(s)}{D(-s)}$. In the case of factorization, $D(s)$ has zeros on the left half-plane from (51).

Substituting $\Phi(s)$ from (50) into (49), then for $\omega(s)$ we get:

$$\omega(s) = \frac{B_0(s)P(s) + D(s)}{B_0(s)}, \tag{52}$$

i.e.

$$\omega_0(s) = B_0(s), \; \omega_1(s) = B_0(s)P(s) + D(s). \tag{53}$$

Let's prove that the closed system (46), (48) is asymptotically stable. Substituting $\omega_0(s), \;\omega_1(s)$ from (52), (53) into

$$\det \begin{bmatrix} P(s) & -M(s) \\ \omega_1(s) & \omega_0(s) \end{bmatrix} = D(s),$$

we obtain that the closed system is asymptotically stable.

Another parameterization, the so-called Youla-Kucera-Desoer [25,35,74], unlike [38,42,43], suggest choosing $\alpha(s)$, $\beta(s)$ will satisfy the following

$$P(s)\beta(s) + M\alpha(s) = 1$$

Diophantine equation which is a special case of Larin's parametrization [10,38,42,43].

4.3. **Asymptotic method.** For a sufficiently large $m >> 1$ as in (10) we accept $\varepsilon = \frac{1}{m}$ and we have equation (23) which includes ε, where, as in (10), (11), and (39), we have

$$D^{\frac{1}{q}} z(t) = (A_0 + \varepsilon A_1) z(t) + \varepsilon G u(t).$$

Then looking for

$$T = T^{(0)} + \varepsilon T^{(1)} = \begin{bmatrix} \hat{T}_1^{(0)} + \varepsilon \hat{T}_1^{(1)} & \hat{T}_2^{(0)} + \varepsilon \hat{T}_2^{(1)} \\ \hat{T}_3^{(0)} + \varepsilon \hat{T}_3^{(1)} & \hat{T}_4^{(0)} + \varepsilon \hat{T}_4^{(1)} \end{bmatrix}, \quad R = \frac{1}{\varepsilon} \hat{R}.$$

For K from (40) we have the following expression

$$K = -\hat{R}^{-1} G' [\hat{T}_3^{(0)} \hat{T}_1^{(0)^{-1}} + \varepsilon(-\hat{T}_3^{(0)} T_1^{(0)^{-1}} T_1^{(1)} T_1^{(0)^{-1}} + T_3^{(1)} T_1^{(0)^{-1}})],$$

where $\hat{T}_3^{(0)}$, $\hat{T}_1^{(0)}$, $T_1^{(1)}$, $T_3^{(1)}$ are defined from [20].

Now, using a particular method, we can redistribute $\omega_0(s)$, $\omega_1(s)$ (48), when equation (43) for a sufficiently large $m >> 1$ goes to the form

$$\ddot{y} + \varepsilon a D^{\alpha} y + \varepsilon b y = \varepsilon u(t)$$

and for the given case

$$P = s^2 + \varepsilon a s^{\frac{p}{q}} + \varepsilon b = P_0(s) + \varepsilon P_1(s), \;\; M = \varepsilon, \;\; \psi = y_1,$$

where $P_0(s) = s^2$, $P_1(s) = a s^{\frac{p}{q}} + b$, and as a result of this, $D(s)D(-s)$ at the first approximation with respect to the small parameter ε has the form

$$D(s)D(-s) \approx c\psi_1^2 s^4 - 2ac\varepsilon s^{2+\frac{p}{q}} - 2\psi_1^2 b \varepsilon s^2 + r \tag{54}$$

and further

$$T(s) = -c(s^2 - \varepsilon a s^{\frac{p}{q}} + \varepsilon b)\psi_1^2.$$

Similarly to the previous case, it is possible to determine $\omega_0(s)$, $\omega_1(s)$ at the first approximation with respect to a small parameter ε. Indeed, from (54) we represent $D(s)D(-s)$ in the following form with the notation $\rho = s^{\frac{2}{\varepsilon}}$, $p = 2k$

$$\begin{aligned} D(\rho)D(-\rho) = c\psi_1^2(\rho^{2q} + 0\rho^{2q-1} + ... + 0\rho^{q+k+1} - \tfrac{2a}{\psi_1^2}\varepsilon\rho^{q+k} + 0\rho^{q+k-1} + ... \\ +0\rho^{q+1} - \tfrac{2b}{c}\varepsilon\rho^q + 0\rho^{q-1} + ... + 0\rho + \tfrac{r}{c\psi_1^2}). \end{aligned} \tag{55}$$

Now we factorize the polynomial (55) in the following form

$$D(\rho) = \sqrt{c}\psi_1(\rho^q + (L' + G'\Pi)N),$$

where in this case

$$L=0,\quad G=\begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix},\ N(\rho)=\begin{bmatrix}1\\\rho\\\vdots\\\rho^{q-1}\end{bmatrix},$$

and the matrix Π is a positive definite solution to the following matrix algebraic Riccati equation [12,21,37,39-41]

$$\Pi F+F'\Pi-(\Pi G+L)(G'\Pi+L')+R=0. \tag{56}$$

Here $F,\ R$ are block diagonal matrices respectively

$$F=\begin{bmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\\dots&\dots&\dots&\dots&\dots\\0&0&0&\dots&1\\0&0&0&\dots&0\end{bmatrix},\quad R=R_0+R_1'+R_1,$$

$$R_0=diag\left[r,0,...,0,-2ac\varepsilon,0,...,0\right],$$

$R_1=diag\left[0,...,0,\frac{b}{c}\varepsilon,...,0\right],\ F,\ R=\tilde{R}_0+\varepsilon\tilde{R}_1,\ \tilde{R}_0=diag\left[r,0,...,0\right],$
$\tilde{R}_1=diag\left[0,...,-rac,0,...,\frac{b}{c},0,...,0\right]$. Choosing such a solution Π, the equation (56) so that the matrices

$$F-G(L'+G'\Pi) \tag{57}$$

are Hurwitz, i.e. the eigenvalues of (57) lie in the left-half plane.

Now we are looking for (56) in the form

$$\Pi=\Pi_0+\varepsilon\Pi_1.$$

For $\Pi_0,\ \Pi_1$we have the following

$$\Pi_0F+F'\Pi_0-\Pi_0GG'\Pi_0+\tilde{R}_0=0, \tag{58}$$

$$\Pi_1(F-GG'\Pi_0)+(F'-\Pi_0GG')\Pi_1+\tilde{R}_1=0, \tag{59}$$

i.e. solving Riccati equation (58) with respect to Π_0 we find a solution so that the matrix $F-GG'\Pi_0$ is Hurwitz. Further solving the Lyapunov equation (59) we get $\Pi_1\geq 0$ and we restore the factorized polynomial [9,34] $D(s)$ in the following form

$$D(s)=D_0(s)+\varepsilon D_1(s),$$

where

$$D_0(s)=\sqrt{c}\psi_1(s^2+G'\Pi_0N(s)),\ D_1(s)=\sqrt{c}\psi_1G'\Pi N(s),$$
$$N\left(s\right)=\left[1,\ s^{2/q},\quad s^{2\left(1-\frac{1}{q}\right)}\right]'.$$

We can easily show that in this case from (51) has the representation

$$T(s)=-\psi_1^2cs^2+\varepsilon(as^{\frac{p}{q}}-b)c\psi_1^2.$$

Then from (51) $B_0(s)=-\psi_1\sqrt{c}$ and therefore from (53)

$$\omega_0(s)=-\psi_1\sqrt{c},$$

$$\omega_1(s) = (-\psi_1\sqrt{c}P_0(s) + D_0(s)) + \varepsilon(-\psi_1\sqrt{c}P_1(s) + D_1(s)).$$

Now we can easily prove that the corresponding closed system is asymptotically stable equal to $D_0(s) + \varepsilon D_1(s)$ which is equal in the first approximation to Hurwitz polynomial $D(s)$.

5. Discretization of the problem (2), (3)

5.1. **Reduction the Cauchy problem (2), (3) to the Volterra integral equation of the second kind with respect to y''.** We consider the following initial problem for a second-order ordinary linear differential equation with constant coefficients and fractional derivatives [1,7,18,51,52,53,59] in subordinate terms, i.e.

$$y''(t) + aD^\alpha y(t) + by(t) = f(t), \quad t > 0, \;\; \alpha \in (1,\, 2), \tag{60}$$

$$\begin{cases} y(0) = 0, \\ y'(0) = y_{10}, \end{cases} \tag{61}$$

where the coefficients $a,\ b$ of the equation (60) and initial data y_{10} - given real numbers, the right-hand part of the equation (60)- given continuous real-valued function, $y(t)$ - the required function. The initial problem (60), (61) is reduced to the Volterra integral equation of the second kind with respect $y\prime\prime(t)$.

Let's perform the following transformation:

$$D^\alpha y(t) = D^{\alpha-1}Dy(t) = DD^{\alpha-1}y(t),$$

where $D = \frac{d}{dt},\;\; \alpha - 1 \in (0,\, 1)$.

Then

$$D^{\alpha-1}y(t) = D\int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y(\tau)d\tau, \tag{62}$$

and

$$D^{\alpha}y(t) = D^2\int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y(\tau)d\tau. \tag{63}$$

Now we transform the integral in (63) so that the differentiation can be introduced under the integral sign:

$$\int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y(\tau)d\tau = -\int\limits_0^t y(\tau)d_\tau\frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!}, \tag{64}$$

considering $2 - \alpha > 0$, we integrate the integral in (64) by parts:

$$\begin{aligned} &-\int\limits_0^t y(\tau)d_\tau\tfrac{(t-\tau)^{2-\alpha}}{(2-\alpha)!} = -\left.\tfrac{(t-\tau)^{2-\alpha}}{(2-\alpha)!}y(\tau)\right|_{\tau=0}^t + \int\limits_0^t \tfrac{(t-\tau)^{2-\alpha}}{(2-\alpha)!}y'(\tau)d\tau = \\ &= \tfrac{t^{2-\alpha}}{(2-\alpha)!}y(0) + \int\limits_0^t \tfrac{(t-\tau)^{2-\alpha}}{(2-\alpha)!}y'(\tau)d\tau = \int\limits_0^t \tfrac{(t-\tau)^{2-\alpha}}{(2-\alpha)!}y'(\tau)d\tau, \end{aligned} \tag{65}$$

for obtaining (65), we take into account the first condition from (61). Then from (62) we get:

$$D^{\alpha-1}y(t) = D\int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y(\tau)d\tau = D\int\limits_0^t \frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!}y'(\tau)d\tau = \int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y'(\tau)d\tau. \tag{66}$$

In the obtained integral (66) we will carry out the above operation again, i.e.

$$\int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!} y'(\tau)d\tau = -\int\limits_0^t y'(\tau)d_\tau \frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!} = -\frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!} y'(\tau)\Big|_{\tau=0}^t +$$
$$+\int\limits_0^t \frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!} y''(\tau)d\tau =$$
$$= \frac{t^{2-\alpha}}{(2-\alpha)!} y'(0) + \int\limits_0^t \frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!} y''(\tau)d\tau = \frac{t^{2-\alpha}}{(2-\alpha)!} y_{10} + \int\limits_0^t \frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!} y''(\tau)d\tau.$$

Thus from (63) we get:

$$\begin{aligned} D^\alpha y(t) &= DD^{\alpha-1}y(t) = D\left[\frac{t^{2-\alpha}}{(2-\alpha)!}y_{10} + \int\limits_0^t \frac{(t-\tau)^{2-\alpha}}{(2-\alpha)!}y''(\tau)d\tau\right] = \\ &= \frac{t^{1-\alpha}}{(1-\alpha)!}y_{10} + \int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y''(\tau)d\tau. \end{aligned} \tag{67}$$

Now we consider the third term on the left-hand side of (60):

$$\begin{aligned} y(t) &= \int\limits_0^t y'(\tau)d\tau + y(0) = \int\limits_0^t y'(\tau)d\tau = \int\limits_0^t d\tau \left\{\int\limits_0^\tau y''(\eta)d\eta + y'(0)\right\} = \\ &= \int\limits_0^t d\tau \int\limits_0^\tau y''(\eta)d\eta + \int\limits_0^t y_{10}d\tau = \\ &= \int\limits_0^t y''(\eta)d\eta \int\limits_\eta^t d\tau + y_{10}t = \int\limits_0^t (t-\tau)y''(\tau)d\tau + y_{10}t. \end{aligned} \tag{68}$$

Now considering (67) and (68), the equation (60) takes the form: $y''(t)+a\left[\frac{t^{1-\alpha}}{(1-\alpha)!}y_{10} + \int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y''(\tau)d\tau\right]+$ $b\left[\int\limits_0^t (t-\tau)y''(\tau)d\tau + y_{10}t\right] = f(t),$
or

$$y''(t) + a\int\limits_0^t \frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!}y''(\tau)d\tau + b\int\limits_0^t (t-\tau)y''(\tau)d\tau = f(t) - a\frac{t^{1-\alpha}}{(1-\alpha)!}y_{10} - by_{10}t,$$

or

$$y''(t) + \int\limits_0^t K_\alpha(t-\tau)y''(\tau)d\tau = F(t), \quad t>0, \tag{69}$$

where

$$K_\alpha(t-\tau) = a\frac{(t-\tau)^{1-\alpha}}{(1-\alpha)!} + b(t-\tau),$$
$$F(t) = f(t) - \left[a\frac{t^{1-\alpha}}{(1-\alpha)!} + bt\right]y_{10}.$$

Here we will discretize the Volterra integral equation of the second kind (69).

5.2. **Discretization.** Considering $y(t) = y_n,\ \ F(t) = F_n$ and

$$y''(t) \approx \frac{y_{n+2} - 2y_{n+1} + y_n}{h^2},$$

we obtain:

$$\frac{y_{n+2} - 2y_{n+1} + y_n}{h^2} + h\sum_{k=0}^{n-1} K_\alpha(t_n - t_k)\frac{y_{k+2} - 2y_{k+1} + y_k}{h^2} = F_n, \quad n \geq 0,$$

where h-step, $t=t_n,\ t_0=0,\ \tau=t_k=kh.$

Then the obtained system of algebraic equations will take the form [31,36,37]:

$$\begin{gathered}y_{n+2}=2y_{n+1}-y_n-h\sum_{k=0}^{n-1}\left[a\tfrac{(t_n-t_k)^{1-\alpha}}{(1-\alpha)!}+b(t_n-t_k)\right](y_{k+2}-2y_{k+1}+y_k)+\\ +h^2\left\{f_n-\left[a\tfrac{t_n^{1-\alpha}}{(1-\alpha)!}+bt_n\right]y_{10}\right\},\quad n\geq 0,\ \ f_n=f(t_n),\ f_0=f(0).\end{gathered}\tag{70}$$

5.3. **The case of even indices.** Representing (70) for an even and an odd indices, we obtain the following pair of relations:

$$\begin{aligned}&y_{2m}=2y_{2m-1}-y_{2m-2}-h\sum_{k=0}^{2m-3}\left[a\tfrac{(t_{2m-2}-t_k)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_k)\right](y_{k+2}-2y_{k+1}+y_k)+\\ &+h^2\left\{f_{2m-2}-\left[a\tfrac{t_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bt_{2m-2}\right]y_{10}\right\},\quad m\geq 1,\end{aligned}$$

$$\begin{aligned}y_{2m}=&\left\{2-h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-3})\right]\right\}y_{2m-1}+\\ &+\left\{-1-h\left[a\tfrac{(t_{2m-2}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-4})\right]+\right.\\ &\left.+2h\left[a\tfrac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-3})\right]\right\}y_{2m-2}+\\ &+\{-h\left[a\tfrac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-3})\right]-\\ &-h\left[a\tfrac{(t_{2m-2}-t_{2m-5})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-5})\right]+\\ &+2h\left[a\tfrac{(t_{2m-2}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-4})\right]\Big\}y_{2m-3}+\\ &\quad+\sum_{k=2}^{2m-4}\left\{-h\left[a\tfrac{(t_{2m-2}-t_k)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_k)\right]+\right.\\ &\quad+2h\left[a\tfrac{(t_{2m-2}-t_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{k-1})\right]-\\ &\quad\left.-h\left[a\tfrac{(t_{2m-2}-t_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{k-2})\right]\right\}y_k+\\ &\quad+\left\{2h\left[a\tfrac{(t_{2m-2}-t_0)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_0)\right]-\right.\\ &\quad\left.-h\left[a\tfrac{(t_{2m-2}-t_1)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_1)\right]\right\}y_1+\\ &\quad+\left\{-h\left[a\tfrac{(t_{2m-2}-t_0)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_0)\right]\right\}y_0+\\ &+h^2\left\{f_{2m-2}-\left[a\frac{t_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bt_{2m-2}\right]y_{10}\right\},\quad m\geq 1.\end{aligned}\tag{71}$$

Thus (71) is a discrete version of problem (2), (3) when the index is even.

5.4. **Odd indices.** Now we consider the case when the indices are odd, i.e.

$$\begin{aligned}&y_{2m+1}=2y_{2m}-y_{2m-1}-\\ &-h\sum_{k=0}^{2m-2}\left[a\tfrac{(t_{2m-1}-t_k)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_k)\right](y_{k+2}-2y_{k+1}+y_k)+\\ &+h^2\left\{f_{2m-1}-\left[a\tfrac{t_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bt_{2m-1}\right]y_{10}\right\},\quad m\geq 1.\end{aligned}$$

Similarly to (71), we group the resulting expression as follows:

$$y_{2m+1}=\left\{\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times\right.$$

$$\times\left(2-h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-3})\right]\right)-$$

$$-1-h\left[a\frac{(t_{2m-1}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-3})\right]+$$

$$+2h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\Bigg\}\,y_{2m-1}+$$

$$+\Bigg\{\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times$$

$$\times\left(-1-h\left[a\frac{(t_{2m-2}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-4})\right]+\right.$$

$$\left.+2h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-3})\right]\right)-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-4})\right]+$$

$$+2h\left[a\frac{(t_{2m-1}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-3})\right]-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\Bigg\}\,y_{2m-2}+$$

$$+\Bigg\{\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times$$

$$\times\left(-h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-3})\right]-\right.$$

$$-h\left[a\frac{(t_{2m-2}-t_{2m-5})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-5})\right]+$$

$$\left.+2h\left[a\frac{(t_{2m-2}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2m-4})\right]\right)-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2m-5})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-5})\right]+$$

$$+2h\left[a\frac{(t_{2m-1}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-4})\right]-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-3})\right]\Bigg\}\,y_{2m-3}+$$

$$+\sum_{k=2}^{2m-4}\left\{2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right\}\times$$

$$\times\left(-h\left[a\frac{(t_{2m-2}-t_k)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_k)\right]+\right.$$

$$+2h\left[a\frac{(t_{2m-2}-t_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{k-1})\right]-$$

$$\left.-h\left[a\frac{(t_{2m-2}-t_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{k-2})\right]\right)-$$

$$-h\left[a\frac{(t_{2m-1}-t_{k-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{k-2})\right]+$$

$$+2h\left[a\frac{(t_{2m-1}-t_{k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{k-1})\right]-$$

$$-h\left[a\frac{(t_{2m-1}-t_{k})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{k})\right]\Bigg\}\,y_k+$$

$$+\left\{\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times\right.$$

$$\times\left(2h\left[a\frac{(t_{2m-2}-t_{0})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{0})\right]-\right.$$

$$\left.-h\left[a\frac{(t_{2m-2}-t_{1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{1})\right]\right)+$$

$$+2h\left[a\frac{(t_{2m-1}-t_{0})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{0})\right]-$$

$$\left.-h\left[a\frac{(t_{2m-1}-t_{1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{1})\right]\right\}y_1+$$

$$+\left\{\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times\right.$$

$$\times\left(-h\left[a\frac{(t_{2m-2}-t_{0})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{0})\right]\right)-$$

$$\left.-h\left[a\frac{(t_{2m-1}-t_{0})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{0})\right]\right\}y_0+$$

$$+\left\{h^2\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times\right.$$

$$\times\left(f_{2m-2}-\left[a\frac{t_{2m-2}^{1-\alpha}}{(1-\alpha)!}+bt_{2m-2}\right]y_{10}\right)+$$

$$\left.+h^2\left(f_{2m-1}-\left[a\frac{t_{2m-1}^{1-\alpha}}{(1-\alpha)!}+bt_{2m-1}\right]y_{10}\right)\right\},\ m\geq 1. \tag{72}$$

Thus, (72) is an odd variant of the discretization of the Cauchy problem (2), (3).

5.5. **Combining the discretization of even and odd cases as a system.** We take the following notation:

$$w_k = \begin{pmatrix} y_{2k} \\ y_{2k+1} \end{pmatrix}, \ k \geq 0. \tag{73}$$

Then combining (71) and (72), taking into account (73), we obtain the following representation for the system of algebraic equations:

$$w_m = \sum_{k=0}^{m-1} A^{(k)} w_k + F_m, \ \ m \geq 1, \tag{74}$$

where

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{pmatrix}, \ \ k \geq 0, \ \ F_m = \begin{pmatrix} F_{m1} \\ F_{m2} \end{pmatrix},$$

$$A_{11}^{(m-1)} = -1 - h\left[a\frac{(t_{2m-2}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_{2m-4})\right] +$$

$$+2h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_{2m-3})\right],$$

$$A_{12}^{(m-1)} = 2 - h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_{2m-3})\right],$$

$$A_{21}^{(m-1)} = \left(2 - h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-2})\right]\right) \times$$

$$\times\left(-1 - h\left[a\frac{(t_{2m-2}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_{2m-4})\right] +\right.$$

$$\left.+2h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_{2m-3})\right]\right) -$$

$$-h\left[a\frac{(t_{2m-1}-t_{2m-4})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-4})\right] +$$

$$+2h\left[a\frac{(t_{2m-1}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-3})\right] -$$

$$-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-2})\right],$$

$$A_{22}^{(m-1)} = \left(2 - h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-2})\right]\right) \times$$

$$\times\left(2 - h\left[a\frac{(t_{2m-2}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_{2m-3})\right]\right) -$$

$$-1 - h\left[a\frac{(t_{2m-1}-t_{2m-3})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-3})\right] +$$

$$+2h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-2})\right]$$

$$A_{11}^{(k)} = -h\left[a\frac{(t_{2m-2}-t_{2k})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_{2k})\right] +$$

$$+2h\left[a\frac{(t_{2m-2}-t_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k-1})\right]-$$

$$-h\left[a\frac{(t_{2m-2}-t_{2k-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k-2})\right],\quad k=\overline{1,m-2},$$

$$A_{12}^{(k)}=-h\left[a\frac{(t_{2m-2}-t_{2k+1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k+1})\right]+$$

$$+2h\left[a\frac{(t_{2m-2}-t_{2k})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k})\right]-$$

$$-h\left[a\frac{(t_{2m-2}-t_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k-1})\right],\quad k=\overline{1,m-2},$$

$$A_{21}^{(k)}=\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times$$

$$\times\left(-h\left[a\frac{(t_{2m-2}-t_{2k})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k})\right]+\right.$$

$$+2h\left[a\frac{(t_{2m-2}-t_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k-1})\right]-$$

$$\left.-h\left[a\frac{(t_{2m-2}-t_{2k-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k-2})\right]\right)-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2k-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2k-2})\right]+$$

$$+2h\left[a\frac{(t_{2m-1}-t_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2k-1})\right]-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2k})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2k})\right],\quad k=\overline{1,m-2},$$

$$A_{22}^{(k)}=\left(2-h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2m-2})\right]\right)\times$$

$$\times\left(-h\left[a\frac{(t_{2m-2}-t_{2k+1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k+1})\right]+\right.$$

$$+2h\left[a\frac{(t_{2m-2}-t_{2k})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_{2k})\right]-$$

$$\left.-h\left[a\frac{(t_{2m-2}-t^{2k-2})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t^{2k-2})\right]\right)-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2k-1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2k-1})\right]+$$

$$+2h\left[a\frac{(t_{2m-1}-t_{2k})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2k})\right]-$$

$$-h\left[a\frac{(t_{2m-1}-t_{2k+1})^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-1}-t_{2k+1})\right],\quad k=\overline{1,m-2},$$

$$A_{11}^{(0)}=-h\left[a\frac{(t_{2m-2}-t_0)^{1-\alpha}}{(1-\alpha)!}+b(t_{2m-2}-t_0)\right],$$

$$A_{12}^{(0)} = 2h\left[a\frac{(t_{2m-2}-t_0)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_0)\right] -$$
$$-h\left[a\frac{(t_{2m-2}-t_1)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_1)\right],$$
$$A_{21}^{(0)} = \left(2 - h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-2})\right]\right) \times$$
$$\times\left(-h\left[a\frac{(t_{2m-2}-t_0)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_0)\right]\right) -$$
$$-h\left[a\frac{(t_{2m-1}-t_0)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_0)\right],$$
$$A_{22}^{(0)} = \left(2 - h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-2})\right]\right) \times$$
$$\times\left(2h\left[a\frac{(t_{2m-2}-t_0)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_0)\right] -\right.$$
$$\left.- h\left[a\frac{(t_{2m-2}-t_1)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-2}-t_1)\right]\right) +$$
$$+2h\left[a\frac{(t_{2m-1}-t_0)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_0)\right] -$$
$$-h\left[a\frac{(t_{2m-1}-t_1)^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_1)\right],$$
$$F_{m1} = h^2\left\{f_{2m-2} - \left[a\frac{t_{2m-2}^{1-\alpha}}{(1-\alpha)!} + bt_{2m-2}\right]y_{10}\right\},\ \ m \geq 1.$$
$$F_{m2} = h^2\left(2 - h\left[a\frac{(t_{2m-1}-t_{2m-2})^{1-\alpha}}{(1-\alpha)!} + b(t_{2m-1}-t_{2m-2})\right]\right) \times$$
$$\times\left(f_{2m-2} - \left[a\frac{t_{2m-2}^{1-\alpha}}{(1-\alpha)!} + bt_{2m-2}\right]y_{10}\right) +$$
$$+h^2\left(f_{2m-1} - \left[a\frac{t_{2m-1}^{1-\alpha}}{(1-\alpha)!} + bt_{2m-1}\right]y_{10}\right),\ \ m \geq 1.$$

5.6. **A classical analogue of the discretization of the problem (2), (3).** Now (74) can be represented in the form:

$$w_m = A^{(m-1)}w_{m-1} + \sum_{k=0}^{m-2} A^{(k)}w_k + F_m,\quad m \geq 1. \tag{75}$$

Expressing the sum in (75) all terms through w_0, we have:

$$w_m = A^{(m-1)}w_{m-1} + \left[1 + \sum_{j=1}^{m-2}\ \sum_{m-2\geq i_j > i_{j-1} > \ldots > i_2 > i_1 \geq 1}\ \prod_{k=1}^{j} A^{(i_k)}\right] A^{(0)}w_0 +$$
$$+\sum_{s=2}^{m-1}\left[1 + \sum_{j=s}^{m-2}\ \sum_{m-2\geq i_j > i_{j-1} > \ldots > i_s \geq s}\ \prod_{k=s}^{j} A^{(i_k)}\right] A^{(s-1)}F_{s-1} + F_m,\quad m \geq 2, \tag{76}$$

$$w_1 = A^{(0)} w_0 + F_1.$$

Finally, solving (76), i.e. excluding also w_{m-1}, we get:

$$w_m = \left[1 + \sum_{j=1}^{m-1} \sum_{m-1 \geq i_j > i_{j-1} > \ldots > i_2 > i_1 \geq 1} \prod_{k=1}^{j} A^{(i_k)}\right] A^{(0)} w_0 +$$

$$+ \sum_{s=2}^{m} \left[1 + \sum_{j=s}^{m-1} \sum_{m-1 \geq i_j > i_{j-1} > \ldots > i_s \geq s} \prod_{k=s}^{j} A^{(i_k)}\right] A^{(s-1)} F_{s-1} + F_m, \quad m \geq 1.$$

Thus (75) is a discretization of the Cauchy problem (2), (3) with a difference of one time step between (76).

6. Conclusions

For the first time, an inverse problem of the third generation is presented for determining the order of the derivative of the subordinate term of the differential equation of oscillatory systems with liquid dampers (DEOSLD). Methods for solving the DEOSLD with nonlocal boundary conditions are proposed. For a sufficiently large mass, an asymptotic method is constructed. It is noted that this method can be useful for constructing programmed trajectories and controls for oscillatory systems with liquid dampers. The algorithm for constructing optimal controllers with the Letov time method and Larin's frequency parameterization method is given. An asymptotic method is also presented in the first approximation of constructing controllers. Finally, a method for discretizing the DEOSLD is proposed, which, in contrast to the classical case, has nonstationary linear equations.

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