NEW REPRESENTATION OF THE MITTAG-LEFFLER FUNCTION THROUGH THE EXPONENTIAL FUNCTIONS WITH RATIONAL DERIVATIVES

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Abstract. In this paper the Mittag-Leffler function is given through the exponential functions for any rational derivatives of $^{m}/_{n}$ order, where $m < n,\ \ n > 1$ are natural irreducible numbers (if $n = 1$ then m is also equal to unity). Unlike the previous papers the given formulas do not contain integrals.

1. Introduction

In recent years, more attention has been focused to the construction of solutions to the Cauchy problem and boundary value problems for fractional order differential equations [4–6]. The obtained results in this area are purely theoretical, i.e. or the existence or uniqueness of solutions of the corresponding problems are proved [7]. In [1–3] the analytical formula is given for solving the Cauchy problem for the fractional order linear differential equations with constant coefficients.

In this paper it is shown that the Mittag-Leffler function can be represented through an exponential function for the general case, i.e., when the order of the derivative is any regular rational number. This scheme allows to investigate the solution of the problem for fractional order differential equations, the step (derivatives) of which is less than one.

2. Some transformations for the Mittag-Leffler function with a shear

As is known [4], the Mittag-Leffler function with a shear can be represented in the following form

$$h_{\frac{1}{n}}(x,\rho) \ = \sum_{k=0}^{\infty} \rho^k \frac{x^{-1+\frac{k+1}{n}}}{\left(-1+\frac{k+1}{n}\right)!}, \quad x \geq x_0 > 0, \tag{1}$$

where n is natural number, x is real argument, $n \in N,\ \ x \in R,\ \ x \geq x_0 > 0,\ \ \rho \in C$- parameter.

It is easy to see from (1) that

$$D^{\frac{1}{n}} h_{\frac{1}{n}}(x,\rho) \ = \rho h_{\frac{1}{n}}(x,\rho). \tag{2}$$

Now we present the function (1) in the following form

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$$
\begin{aligned}
& h_{\frac{1}{n}}(x,\rho) = \frac{x^{-1+\frac{1}{n}}}{\left(-1+\frac{1}{n}\right)!} + \rho\frac{x^{-1+\frac{2}{n}}}{\left(-1+\frac{2}{n}\right)!} + \cdots + \rho^{n-1}\frac{x^0}{0!} + \\
& +\rho^n\frac{x^{\frac{1}{n}}}{\left(\frac{1}{n}\right)!} + \rho^{n+1}\frac{x^{\frac{2}{n}}}{\left(\frac{2}{n}\right)!} + \cdots + \rho^{2n-1}\frac{x}{1!} + \\
& +\rho^{2n}\frac{x^{1+\frac{1}{n}}}{\left(1+\frac{1}{n}\right)!} + \rho^{2n+1}\frac{x^{1+\frac{2}{n}}}{\left(1+\frac{2}{n}\right)!} + \cdots + \rho^{3n-1}\frac{x^2}{1!} + \cdots + \\
& +\rho^{sn}\frac{x^{s-1+\frac{1}{n}}}{\left(s-1+\frac{1}{n}\right)!} + \rho^{sn+1}\frac{x^{s-1+\frac{2}{n}}}{\left(s-1+\frac{2}{n}\right)!} + \cdots + \rho^{(s+1)n-1}\frac{x^s}{s!} + \\
& + \ldots \quad \ldots \ \ldots \quad \ldots \ \ldots \quad \ldots \ \ldots \quad \ldots \ \ldots \quad \ldots
\end{aligned} \tag{3}
$$

Summing (3) over the corresponding columns, we get:

$$
h_{\frac{1}{n}}(x,\rho) \ = J_0(x,\rho^n) + \rho J_1(x,\rho^n) + \cdots + \rho^{n-1}J_{n-1}(x,\rho^n), \tag{4}
$$

where

$$
J_s(x,\rho^n) = \sum_{k=0}^{\infty}\rho^{kn}\frac{x^{-1+k+\frac{s+1}{n}}}{\left(-1+k+\frac{s+1}{n}\right)!}, \quad s=\overline{0,n-1}. \tag{5}
$$

The formula (4) enable to relate the Mittag-Leffler function with a shear via an exponential function. For this, it is sufficient to reduce the function $J_s(x,\rho^n)$ from (5) to an exponential function.

3. Reduction the function $J_s(x,\rho^n)$ to an exponential function

Now consider the fractional derivative order $\frac{s+1}{n}$of the function $J_s(x,\rho^n)$ from (5)

$$
D^{\frac{s+1}{n}}J_s(x,\rho^n) == \rho^n\left(\frac{x^0}{0!} + \rho^n\frac{x}{1!} + \rho^{2n}\frac{x^2}{2!} + \cdots\right) = \rho^n e^{\rho^n x}, \tag{6}
$$

where, when obtaining (6), we took into account that $\frac{x^{-1}}{(-1)!} = \delta(x) = 0$ due to $x \geq x_0 > 0$.

Now we integrate the equality (6) with fractional order $\frac{s+1}{n}$

$$
I_{x_0}^{\frac{s+1}{n}} D_{x_0}^{\frac{s+1}{n}} J_s(x,\rho^n) = \rho^n I_{x_0}^{\frac{s+1}{n}} e^{\rho^n x},
$$

i.e., by the definition of the Riemann-Liouville fractional integral [4], we obtain

$$
J_s(x,\rho^n) = \rho^n\int\limits_{x_0}^{x}\frac{(x-t)^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}e^{\rho^n t}dt + \tilde{J}_s(x,\rho^n), \tag{7}
$$

where

$$
D_{x_0}^{\frac{s+1}{n}}\tilde{J}_s(x,\rho^n) = 0. \tag{8}
$$

It is easy to see that from (8) the function $\tilde{J}_s(x,\rho^n)$ can be determined in the following form:

$$
\tilde{J}_s(x,\rho^n) = e^{\rho^n x_0}\frac{x^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}. \tag{9}
$$

4. Calculation of the integral from (7)

Taking into account (9) in (7), we have:

$$
J_s(x,\rho^n) = \rho^n\int\limits_{x_0}^{x}\frac{(x-t)^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}e^{\rho^n t}dt + e^{\rho^n x_0}\frac{x^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}.
$$

Let us substitution

$$
\begin{aligned}
& J_s(x,\rho^n) = -\rho^n\int_{x-x_0}^{0}\frac{\xi^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}e^{\rho^n(x-\xi)}d\xi + e^{\rho^n x_0}\frac{x^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!} = \\
& = \rho^n e^{\rho^n x}\frac{1}{\left(\frac{s+1}{n}-1\right)!}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\rho^{nm}\frac{(x-x_0)^{\frac{s+1}{n}+m}}{\frac{s+1}{n}+m} + e^{\rho^n x_0}\frac{x^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}.
\end{aligned} \tag{10}
$$

Substituting (10) into (4) for $h_{\frac{1}{n}}(x,\rho)$ we have

$$\begin{gathered} h_{\frac{1}{n}}(x,\rho)=\sum_{s=0}^{n-1}\rho^{s}J_{s}(x,\rho^{n})=\sum_{s=0}^{n-1}\rho^{s+n}e^{\rho^{n}x}\frac{1}{\left(\frac{s+1}{n}-1\right)!}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\frac{(x-x_{0})^{\frac{s+1}{n}+k}}{\frac{s+1}{n}+k}\rho^{nk}+ \\ +\sum_{s=0}^{n-1}\rho^{s+n}e^{\rho^{n}x_{0}}\frac{x^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}. \end{gathered} \tag{11}$$

Now in (11) we substitute the parameter ρ with a new parameter $\lambda^{\frac{1}{m}}$, i.e., $\rho=\lambda^{\frac{1}{m}}$. Then we have

$$h_{\frac{1}{n}}(x,\rho)=h_{\frac{1}{n}}(x,\lambda^{\frac{1}{m}}). \tag{12}$$

Let $\alpha=\frac{m}{n}$, where m and n natural number, and at $1<m<n$. At $n=1$, then $m=1$.

We calculate the following fractional derivative:

$$D^{\alpha}h_{\frac{1}{n}}(x,\lambda^{\frac{1}{m}})=D^{\frac{m}{n}}h_{\frac{1}{n}}(x,\lambda^{\frac{1}{m}})=\lambda h_{\frac{1}{n}}(x,\lambda^{\frac{1}{m}}), \tag{13}$$

Note that (13) follows from (1) and (2).

Taking into account the substitution $\rho=\lambda^{\frac{1}{m}}$ in (11), we have

$$\begin{gathered} h_{\frac{1}{n}}(x,\lambda^{\frac{1}{m}})=\sum_{s=0}^{n-1}\lambda^{\frac{s+n}{m}}e^{\lambda^{\frac{n}{m}}x}\frac{1}{\left(\frac{s+1}{n}-1\right)!}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\lambda^{\frac{nk}{m}}\frac{(x-x_{0})^{\frac{s+1}{n}+k}}{\frac{s+1}{n}+k}+ \\ +\sum_{s=0}^{n-1}\lambda^{\frac{s}{m}}e^{\lambda^{\frac{n}{m}}x_{0}}\frac{x^{\frac{s+1}{n}-1}}{\left(\frac{s+1}{n}-1\right)!}. \end{gathered} \tag{14}$$

Thus, the Mittag-Leffler function with shear (1) through the exponential function is represented in the form (14). To check the result (14), consider the case, $\alpha=1$, i.e., $n=1$, $m=1$.

Then from (14), we have:

$$\begin{gathered} h_{\frac{1}{n}}(x,\lambda)=\lambda e^{\lambda x}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\lambda^{k}\frac{(x-x_{0})^{1+k}}{1+k}+e^{\lambda x_{0}}= \\ =-e^{\lambda x}\sum_{p=1}^{\infty}\frac{(-1)^{p}}{(p)!}\lambda^{p}(x-x_{0})^{p}+e^{\lambda x_{0}}=-e^{\lambda x}\left(e^{-\lambda(x-x_{0})}-1\right)+e^{\lambda x_{0}}=e^{\lambda x}. \end{gathered}$$

Thus, for $\alpha=1$, the Mittag-Leffler function with a shear turns into an exponential Euler function $e^{\lambda x}$.

Keywords: Mittag-Leffler Function, Exponential Function, Rational Number.

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