Sturm-Liouville, PT-symmetric operators and differential algebraic equations

Carsten Trunk

SOMPATY Lecture

21. October 2021

Overview

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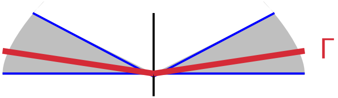

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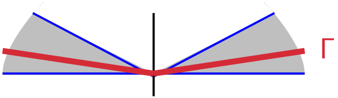

4. Differental algebraic equations: E, A matrices (or operators)

$$E\dot{x} = Ax, \quad \text{in chip re-design: Perturb } E.$$

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$$k(x,y) = \frac{1}{\pi}\begin{cases} (\pi - y)x & \text{if } 0 \le x \le y \le \pi \\ (\pi - x)y & \text{if } 0 \le y \le x \le \pi, \end{cases}$$

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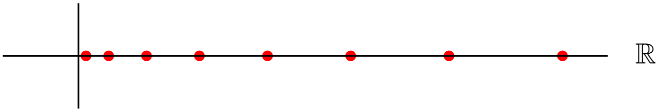

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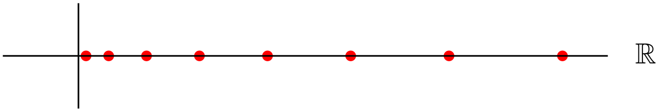

$f \in L^2$:

$$f(x) = \frac{2}{\pi} \sum_{n\in\mathbb{N}} b_n \sin nx = \frac{2}{\pi} \sum_{n\in\mathbb{N}} (f, \sin n\,\cdot)_{L^2} \sin nx \quad \textit{(Fourier series)}$$

where $b_n = \int_0^\pi f(t) \sin nt \, dt$.

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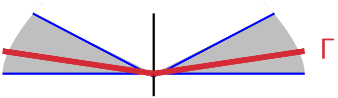

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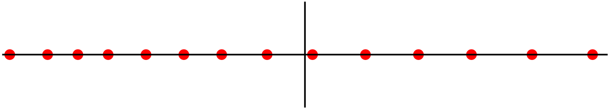
Then the difference of the resolvents is 1-dim and we obtain

$$\sigma(T) = \sigma_p(T) \subset \mathbb{R}.$$



Indefinite Sturm Liouville

Interested in

- the location of point and essential spectrum
- eigenvalue asymptotics
- non-real spectrum
- accumulation of non-real eigenvalues or of eigenvalues in gaps of the essential spectrum

Indefinte Sturm-Liouville, more difficult

Theorem (Math. Ann.'13)

Let $r := \operatorname{sgn}$, $p = 1$, *and* $q \in L^\infty(\mathbb{R})$, $\operatorname{essinf} q < 0$,

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Then the non-real spectrum of T *is contained in*

$$\Big\{ \lambda \in \mathbb{C} : \operatorname{dist}\big(\lambda, (-d, d)\big) \leq 5\|q\|_\infty, \ |\mathrm{Im}\lambda| \leq 2\|q\|_\infty \Big\},$$

where

$$d := -5 essinf_{x\in\mathbb{R}}\, q(x) > 0.$$

Related problems (in other papers):

- $q \in L^1(\mathbb{R})$

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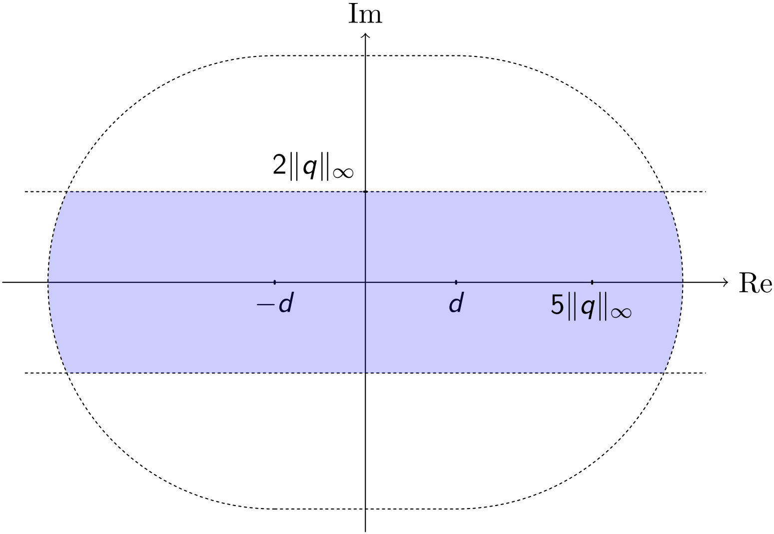
Related problems (in other papers):

- $q \in L^1(\mathbb{R})$
- $q \in L^1_{loc}(\mathbb{R})$
- $q \in L^1_{loc}(\mathbb{R})$ and periodic
- Replace sgn

Indefinte Sturm-Liouville, more difficult II



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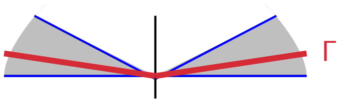

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$$\ddot{z}(t) = -Az(t) - D\dot{z}(t) \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} z(t) \\ x(t) \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & I \\ -A & -D \end{pmatrix}}^{=:\, T} \begin{pmatrix} z(t) \\ x(t) \end{pmatrix}$$

2. Order systems II

Theorem (B. Jacob, C. Tretter, CT, H. Vogt, Math. Meth. Appl. Sci.'18)

$\sigma(T) \subset$

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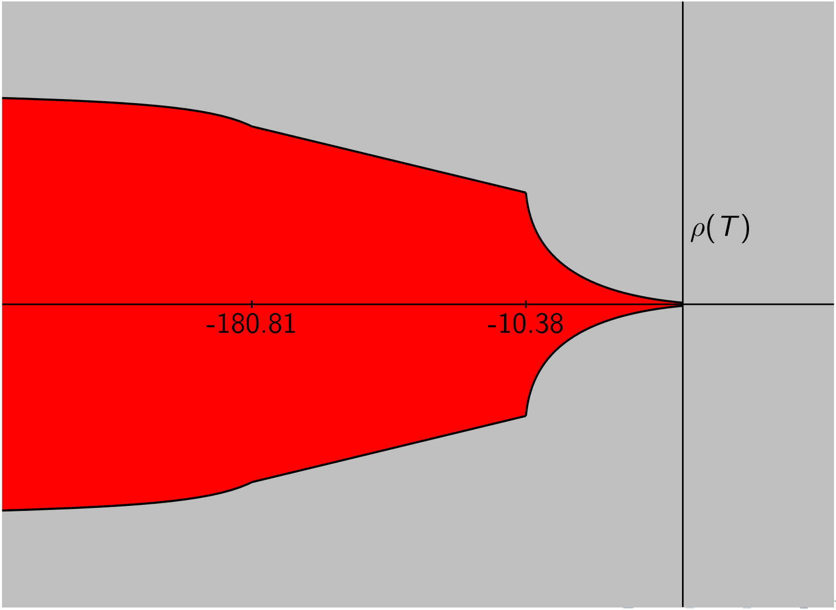
Proof: With the quadratic numeric range.

2. Order systems III



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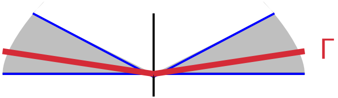

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❸ Non Hermitian QM, Physical Review Letters '98

VOLUME 80, NUMBER 24 PHYSICAL REVIEW LETTERS 15 JUNE 1998

Real Spectra in Non-Hermitian Hamiltonians Having $\mathcal{PT}$ Symmetry

Carl M. Bender[1] and Stefan Boettcher[2,3]
[1]Department of Physics, Washington University, St. Louis, Missouri 63130
[2]Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
[3]CTSPS, Clark Atlanta University, Atlanta, Georgia 30314
(Received 1 December 1997; revised manuscript received 9 April 1998)

The condition of self-adjointness ensures that the eigenvalues of a Hamiltonian are real and bounded below. Replacing this condition by the weaker condition of $\mathcal{PT}$ symmetry, one obtains new infinite classes of complex Hamiltonians whose spectra are also real and positive. These $\mathcal{PT}$ symmetric theories may be viewed as analytic continuations of conventional theories from real to complex phase space. This paper describes the unusual classical and quantum properties of these theories. [S0031-9007(98)06371-6]

The Hamiltonian studied by Bessis is just one example of a huge and remarkable class of non-Hermitian Hamiltonians whose energy levels are real and positive. The purpose of this Letter is to understand the fundamental properties of such a theory by examining the class of quantum-mechanical Hamiltonians

$$H = p^2 + m^2x^2 - (ix)^N \qquad (N \text{ real}). \tag{1}$$

As a function of N and mass m^2 we find various phases with transition points at which entirely real spectra begin to develop complex eigenvalues.

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$$\ell(y) := -y''(z) - (iz)^{N+2}y(z), \quad N \in \mathbb{N}$$

on a wedge shaped contour $z \in \Gamma \subset \mathbb{C}$

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1. Potential is complex valued.
2. Γ.
3. For eigenvalues we need operators. Which? Domains?
4. What is $\mathcal{PT}$ symmetric ? How it will help us?

$\mathcal{PT}$ symmetric expressions

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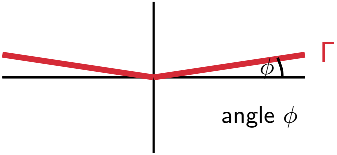

Let $(\mathcal{P}f)(z) = f(-\bar{z})$ parity and $\quad (\mathcal{T}f)(z) = \overline{f(z)}$ time reversal.
We have

$\ell(\mathcal{PT}y(z))$

$\mathcal{PT}$ symmetric expressions

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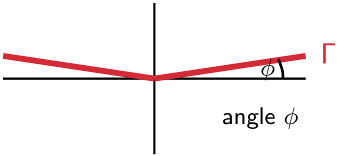

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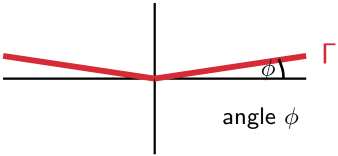

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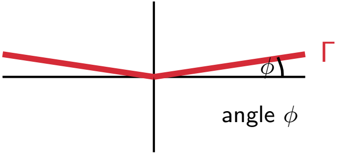

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hence (formally)

$$\ell\mathcal{PT} = \mathcal{PT}\ell$$

and ℓ is called $\mathcal{PT}$ *symmetric*.

Overview

1. Indefinte Sturm-Liouville:

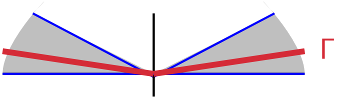
$$Ty := \frac{1}{r}\big(-(py')' + qy\big).$$

2. 2. Order systems:

$$\ddot{u} = -30u_{\xi\xi\xi\xi} - 3\dot{u}_{\xi\xi\xi\xi} - u_{\xi t}.$$

3. Non Hermitian quantum mechanic:

$$Ty(z) := -y''(z) + z^2(iz)^{\epsilon}y(z), \quad \epsilon > 0, \quad z \in \Gamma.$$



4. ❹ Differental algebraic equations (DAE):

$$E\dot{x} = Ax, \quad \text{in chip re-design: Perturb } E.$$

Simple Example

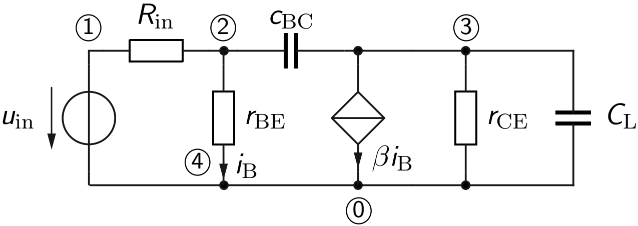
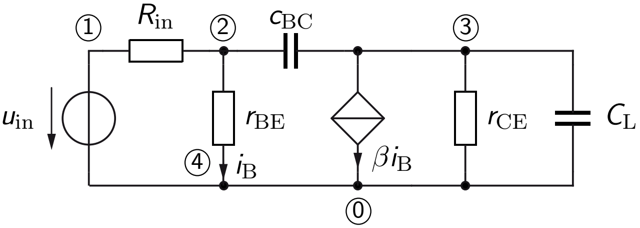


Figure: Amplifier

		Laplace transform
Resistor	$i = \frac{V}{R}$	$i = \frac{V}{R}$

Simple Example



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		Laplace transform
Resistor	$i = \frac{V}{R}$	$i = \frac{V}{R}$
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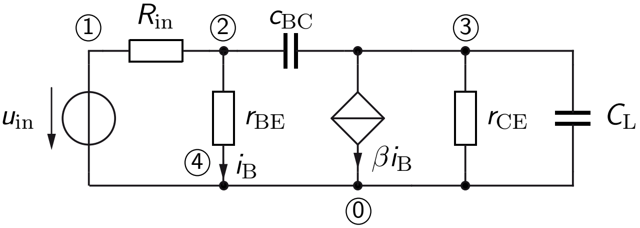

Figure: Amplifier

		Laplace transform
Resistor	$i = \frac{V}{R}$	$i = \frac{V}{R}$
Capacitance	$i = C \cdot \dot{V}$	$i = C \cdot sV$
Current source (current controlled)	$i = \beta i_\beta$	$i = \beta i_\beta$

Simple Example

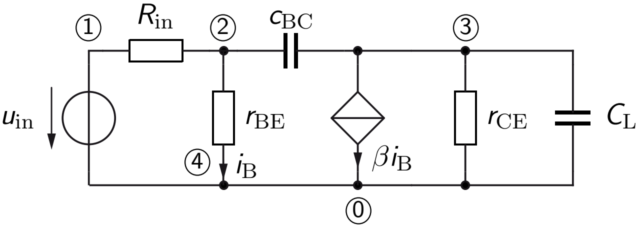

Figure: Amplifier

$$\begin{pmatrix} \frac{1}{R_{\mathrm{in}}} & -\frac{1}{R_{\mathrm{in}}} & 0 & 0 & 1 & 0 \\ -\frac{1}{R_{\mathrm{in}}} & \frac{1}{R_{\mathrm{in}}} + \frac{1}{r_{\mathrm{BE}}} + c_{\mathrm{BC}} s & -c_{\mathrm{BC}} s & -\frac{1}{r_{\mathrm{BE}}} & 0 & 0 \\ 0 & -c_{\mathrm{BC}} s & \frac{1}{r_{\mathrm{CE}}} + c_{\mathrm{BC}} s + C_{\mathrm{L}} s & 0 & 0 & \beta \\ 0 & -\frac{1}{r_{\mathrm{BE}}} & 0 & \frac{1}{r_{\mathrm{BE}}} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ i_{\mathrm{in}} \\ i_{\mathrm{B}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{\mathrm{in}} \\ 0 \end{pmatrix}$$

DAE

$$\underbrace{\begin{pmatrix} \frac{1}{R_{\text{in}}} & -\frac{1}{R_{\text{in}}} & 0 & 0 & 1 & 0 \\ -\frac{1}{R_{\text{in}}} & \frac{1}{R_{\text{in}}} + \frac{1}{r_{\text{BE}}} + c_{\text{BC}} s & -c_{\text{BC}} s & -\frac{1}{r_{\text{BE}}} & 0 & 0 \\ 0 & -c_{\text{BC}} s & \frac{1}{r_{\text{CE}}} + c_{\text{BC}} s + C_{\text{L}} s & 0 & 0 & \beta \\ 0 & -\frac{1}{r_{\text{BE}}} & 0 & \frac{1}{r_{\text{BE}}} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{{\color{red}E} s - {\color{blue}A}} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ i_{\text{in}} \\ i_{\text{B}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_{\text{in}} \\ 0 \end{pmatrix}$$

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$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{\text{BC}} & -c_{\text{BC}} & 0 & 0 & 0 \\ 0 & -c_{\text{BC}} & c_{\text{BC}} + C_{\text{L}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{E} s - \underbrace{\begin{pmatrix} \frac{1}{R_{\text{in}}} & -\frac{1}{R_{\text{in}}} & 0 & 0 & 1 & 0 \\ -\frac{1}{R_{\text{in}}} & \frac{1}{R_{\text{in}}} + \frac{1}{r_{\text{BE}}} & 0 & -\frac{1}{r_{\text{BE}}} & 0 & 0 \\ 0 & 0 & \frac{1}{r_{\text{CE}}} & 0 & 0 & \beta \\ 0 & -\frac{1}{r_{\text{BE}}} & 0 & \frac{1}{r_{\text{BE}}} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{A}$$

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$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{\text{BC}} & -c_{\text{BC}} & 0 & 0 & 0 \\ 0 & -c_{\text{BC}} & c_{\text{BC}} + C_{\text{L}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{E} s - \underbrace{\begin{pmatrix} \frac{1}{R_{\text{in}}} & -\frac{1}{R_{\text{in}}} & 0 & 0 & 1 & 0 \\ -\frac{1}{R_{\text{in}}} & \frac{1}{R_{\text{in}}} + \frac{1}{r_{\text{BE}}} & 0 & -\frac{1}{r_{\text{BE}}} & 0 & 0 \\ 0 & 0 & \frac{1}{r_{\text{CE}}} & 0 & 0 & \beta \\ 0 & -\frac{1}{r_{\text{BE}}} & 0 & \frac{1}{r_{\text{BE}}} & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}}_{A}$$

DAE and transfer function

$$(Es - A)\, v = b$$

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Problem

How to change the modulus of the *transfer function* H on $j\mathbb{R}$

$$H(j\omega) = \frac{D_1 \;\; b}{\det\left(Ej\omega - A\right)}$$

through a new capacitance c?

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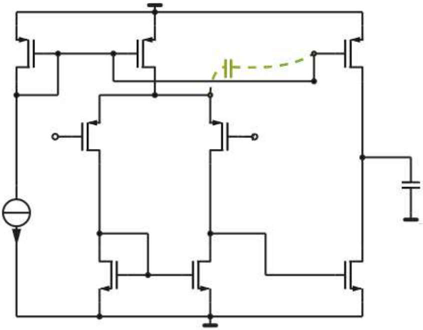
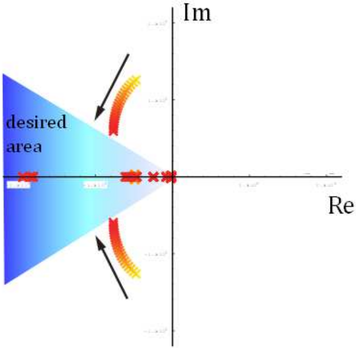
New capacitance:

$$E \to E + cpp^\top \quad \text{with} \quad p = e_j - e_k .$$

Network redesign (Sommer, Krauße and others)

- Method: Add step-by-step capacitances to the network between the nodes i and j, described by the pencil $sc_{ij}(e_i - e_j)(e_i - e_j)^T$, $c_{ij} > 0$.
- New matrix pencil: $s(E + c_{ij}(e_i - e_j)(e_i - e_j)^T) - A$





Thank you.